Racing with a rearview mirror: Outcome lags and investment fluctuations *

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March 6, 2026

Abstract

We model an R&D race in which investors sequentially attempt to achieve an innovation using a risky technology that produces outcomes with a lag. This lag creates a tradeoff between the incentive to invest in a potentially rewarding technology and the risk of being preempted by competitors. In equilibrium, players alternate between periods of strictly decreasing investment and investment breaks, which vanish when the outcome lag is small enough. By contrast, without an outcome lag investment is constant until the common belief that the innovation is feasible reaches a threshold, after which investment stops forever. We thus identify a novel economic force that drives fluctuations in R&D spending. While socially optimal investment is also non-monotonic with an outcome lag, the equilibrium is inefficient. Non-monotonic investment patterns persist when the outcome lag is uncertain, when investment costs are convex, and when investors decide both when and how much to invest.

Keywords: Race for innovation; Outcome lag; Experimentation; Investment fluctuations.

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[*]We are grateful to Christoph Carnehl, Henrique Castro-Piers, Yoon-Koo Che, Louis-Marie Harpedanne, Johannes Hörner, Eugen Kovac, Inés Moreno de Barreda, Guillaume Haeringer, Jan Knoepfle, Nenad Kos, Ignacio Monzon, Pauli Murto, Sven Rady, Régis Renault, Julia Salmi, Aner Sela, José de Sousa, and Yannick Viossat for their insightful suggestions, as well as to the participants at One World Mathematical Game Theory seminar, the Lisbon Meetings in Game Theory and Applications 2023, ESEM conference 2023, SAET conference 2023, University of Vienna, CY, ENS-Paris Saclay, IHP Parisian Game Theory seminar, QMUL-City Economic Theory workshop 2024, Mediterranean Game Theory Symposium 2024, SING conference 2024, Oligo Conference 2024, Université de Nancy, Bocconi University, Essen-Duisburg University, Paris School of Economics, CRETA Economic Theory conference 2025, RITM-Université Paris-Saclay, and HEC Paris.

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1 Introduction

Research and development is a risky business: It involves a race to achieve an innovation, using experimentation technologies that often require substantial time to produce results and, in the best-case scenario, still require numerous trials before yielding a success. The time it takes for an experiment to yield results, i.e., the *outcome lag*, varies depending on the nature of the experiment, but is often significant. For example, the median time from the first human trial to marketing authorization for a new medical drug is estimated to be 7.3 years.[1] Outcome lags can be even longer in fields like physics. In space exploration, missions frequently take over two decades to yield results, not including the time needed to design and build the spacecraft.[2] In astronomy, long development times are needed to build measuring instruments and collect data.

Such outcome lags amplify the risks associated with investment in research and development. A case in point is the 1995 discovery of the first exoplanet by Michel Mayor and his PhD student Didier Queloz, made using the ELODIE spectrograph, which Mayor began developing five years earlier. Mayor and Queloz were not alone in the race to discover exoplanets; several other teams had been scanning the skies since the 1970s. When Mayor began work on ELODIE in 1990, he was thus fully aware that others were also searching, even if none had yet succeeded. To understand the tradeoff faced by potential investors in the presence of an outcome lag, imagine a first-year PhD student in the early 1980's choosing a research topic. One bold option is to join Mayor in the search for exoplanets. Success would guarantee a stellar scientific career. But this path involves two major risks. First, exoplanets might not exist. Second, other teams might be on the verge of making the discovery, preempting the student's contribution. Faced with these two risks—failure and preemption—the student might opt for a safer, if less groundbreaking, research agenda. By 1992, when Queloz began his PhD, the tradeoff had shifted. On the one hand, hopes of finding an exoplanet were lower, as a decade of searching the skies had turned up nothing. On the other hand, the same history had discouraged many competitors. The key question Queloz faced was whether the pessimism induced by past failures dominated the strategic value of reduced competition. As it turned out, Queloz made the right bet. His decision to pursue the exoplanet search led to the historic discovery that earned him and Mayor the 2019 Nobel Prize in Physics.[3]

[1] See Martin et al. (2017) and de Jong et al. (2024).

[2] For example, the New Horizons probe, launched in 2006, passed Pluto in 2015 and is expected to explore the Kuiper Belt in 2025. Similarly, the JUICE probe, launched in 2005 aboard an Ariane 5 rocket, is expected to arrive in Jupiter's orbit in 2031 to study its icy moons.

[3] See Mayor and Queloz (1995).

Despite their obvious relevance, the impact of outcome lags on dynamic investment has been largely overlooked in the microeconomic literature. This paper aims to fill that gap by studying R&D investment over time in settings where firms compete to be the first to achieve an innovation, using technologies that generate outcomes only after a delay. While one could construct a three-period model with a one-period outcome lag that effectively highlights the key trade-offs faced by investors in the exoplanet anecdote, it remains insufficient to capture the full range of possible dynamics induced by outcome lags. We therefore develop an infinite-horizon model that accommodates an arbitrary outcome lag. Besides its immediate findings, our framework also offers a versatile tool that can be applied or extended in future work studying strategic interactions with outcome lags.

Specifically, we analyze a continuous-time model in which a continuum of agents sequentially try to achieve a success by investing a fraction of their unit endowment in a costly technology. The arrival of a success is the only source of payoff, and the game ends after the first success. The technology produces outcomes with a lag Δ, in the sense that the outcome of an investment made at time t, whether a success or a failure, only materializes at $t+\Delta$. Success may or may not be feasible. If it is feasible, its instantaneous Poisson arrival rate is proportional to the amount of investment made Δ time units ago. If success is infeasible, all investment is wasted. Agents start with a common prior, and actions and outcomes are publicly observed, so that they hold a common belief about the feasibility of success at all times. Conditionally on no success arriving, i.e., on the game continuing, players gradually become more pessimistic about the feasibility of success. We analyze two distinct scenarios. First, we consider the case of a single funding agency that coordinates individual investments and receives the payoff in case of success. This problem requires the application of optimal-control techniques with a lagged control variable, which are non-standard. Next, we examine the strategic setting where each agent decides how much to invest by best-responding to the investment decisions of her predecessors and receives the payoff in case of success.[4]

Our main findings are as follows. When there is no outcome lag, both problems yield the same solution: All agents invest their total endowment in the technology until the common belief reaches a critical threshold, at which point investment stops forever. The introduction of an outcome lag kills this cutoff structure. For any positive value of Δ, the optimal investment dynamics become non-monotonic. The funding agency alternates between finite periods of decreasing investment and finite periods of increasing investment. In equilibrium, when Δ is

[4] The agency problem is a *decision problem of experimentation*, as the agency faces no competition and takes into account the impact of its current investment on its future beliefs and payoffs. In contrast, strategic investors compete with each other but *do not experiment*, as they play only once.

sufficiently large, players cycle between phases of positive, strictly decreasing, investment and periods of no investment, which we refer to as *investment breaks*. We show that longer outcome lags lead to longer and more frequent investment breaks. This occurs because the outcome lag effectively creates competition between each player at time t and all players in the interval $[t-\Delta, t)$. Thus as Δ increases, competition intensifies. When Δ is large given players' optimism about the feasibility of success, some investors may choose not to invest if the likelihood of being preempted by a past competitor is too high. As the period of no investment extends, the risk of preemption gradually decreases until investment becomes attractive again. This results in a non-monotonic pattern: For every player who chooses not to invest, there exists a strictly more pessimistic player who does. For small outcome lags, investment breaks disappear. In this case, players continue investing, though at a gradually decreasing rate as their optimism wanes, despite the decreasing risk of preemption. While the total amount of investment is the same in equilibrium and in the funding-agency solution, the equilibrium is always inefficient. The reason is that, unlike strategic investors, the forward-looking funding agency knows that there will be future opportunities for investment. As a result, the agency has fewer incentives to rush investment and can better manage the cost of duplication by decreasing investment during certain periods and increasing it again once the risk of duplication has sufficiently declined. Our results provide a novel, strategic, explanation for the empirically documented fluctuations and breaks in R&D investment.[5]

We analyze three extensions of our model to assess the robustness of our findings.

- In R&D contexts, firms may not know the outcome lag exactly. At best, they may be able to estimate a time window within which the outcome is likely to materialize. To account for this uncertainty, we extend our model by assuming that players only know the distribution of the outcome lag. In the absence of a success, they continuously update their beliefs about both the length of the lag and the feasibility of the technology. We show that, as in the case of a known outcome lag, investment breaks still emerge in equilibrium when the lag is sufficiently long.

- Standard investment models assume convex costs associated with adjusting capital. As Eisner and Strotz (1963) highlighted in their seminal work, under convex costs and divisible investment projects, there is no technological reason to expect frequent episodes of zero investment.[6] We analyze a version of our model with convex investment costs,

[5] See the related-literature section.

[6] A common implication of relaxing the assumption of convex adjustment cost in the neoclassical investment model is that investment may occur in lumps.

and confirm both the latter intuition and the results of our baseline model: There are no investment breaks in equilibrium, but investment remains non-monotonic, provided certain conditions on the outcome lag are met.

- Finally, as Pindyck (1991) noted, a crucial yet often overlooked characteristic of investment expenditures is that they can be delayed. This flexibility allows firms to wait for new information about prices, costs, and other market conditions before committing resources. We extend our model to a setting where agents invest only once but can decide *when and how much* to invest. If several agents achieve success simultaneously, they share the payoff from the innovation equally. We find that when Δ is sufficiently large, equilibrium investment dynamics become extremely lumpy: A positive mass of agents invests every Δ periods until they become too pessimistic, at which point investment stops forever.

Related literature. Our main result is that the fear of preemption caused by outcome lags can give rise to lumpy investment dynamics. Empirical evidence of lumpy investment has been found in US data by Wang and Zhang (2025), who analyze annual panel data between 1968 and 2020, and document that firm-level R&D investments are lumpy, marked by periods of inaction followed by sharp increases in R&D expenditures. Similarly, Nilsen and Schiantarelli (2003) observe investment breaks in Norwegian data, where 21% of their observations report zero equipment investment. Cooper et al. (1999) also provide evidence that the probability of a plant experiencing a large investment episode increases in the time elapsed since the last such episode, suggesting that periods of large investment are often followed by periods of low investment. Doms and Dunne (1998) document the distributions of investment adjustment for a sample of over 13,700 manufacturing plants drawn from over 300 four-digit industries in the US. They find evidence of lumpy investment, i.e., that a large portion of investment at the plant level is concentrated in a few episodes.

We model the innovation technology as a two-armed exponential bandit with conclusive breakthroughs à la Keller, Rady and Cripps (2005), albeit with an outcome lag. Therefore, our model is related to the literature on strategic experimentation, to which we contribute by introducing an outcome lag into the model.[7] While e.g. Bolton and Harris (1999), Keller and

[7]The bandit games analyzed in the literature differ in several dimensions: the type of the risky action ("good-news" vs. "bad-news" models), the news process (which may be conclusive or gradual), as well as the information players have on each other's actions and outcomes. Private information in experimentation games has been studied, among others, by Rosenberg, Solan and Vieille (2007), Bonatti and Hörner (2011), Heidhues, Rady and Strack (2015), and Marlats and Ménager (2021). See Bergemann and Välimäki (2008) for applications of such bandit games in economics and finance, and Hörner and Skrzypacz (2016) for a review of more recent references.

Rady (2010), and Boyarchenko (2019) study models of gradual learning, which may be well-suited for research and development applications, we have chosen the conclusive-news setting for tractability.

Our game is also a race for innovation where players care about being the *first to succeed* as in Das and Klein (2024). This sets our paper apart from the vast literature on competition and learning, initiated by the works of Reinganum (1981) and Fudenberg and Tirole (1985), where players compete to be the *first to invest* (see also Choi (1991), Malueg and Tsutsui (1997), Bobtcheff and Mariotti (2012), Argenziano and Schmidt-Dengler (2014)). Interestingly, some investment-timing papers also find non-monotonic equilibria. The intensity of competition is non-monotonic over time in the preemption game of Bobtcheff and Mariotti (2012), in which two potential competitors invest at random secret times. Argenziano and Schmidt-Dengler (2014) also find that the investment dynamics may not be monotonic in time in a model where the investment cost decreases with time.

Finally, our model is related to the model studied in Gordon, Marlats, and Ménager (2021), where a team of partners work together to achieve a project that is commonly known to be feasible. Players learn immediately whether they succeed but observe their partners' outcomes only after a fixed lag. While the unique symmetric equilibrium in their model also exhibits periodic effort behavior, with alternating phases of maximal and minimal effort, the driving force underlying the non-monotonicity of effort is free-riding, which is absent from our model. In their model, players delay costly investment to see whether one of their partners has already completed the project, while in our model, they delay investment out of fear of being preempted.

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 analyzes the problem of a single funding agency. In Section 4, we characterize the essentially unique equilibrium and show that it is inefficient from the perspective of the funding agency. In Section 5, we explore several variations of the model and conclude in Section 6. Proofs are gathered in the Appendix.

2 The set-up

There is a continuum of agents, indexed by $t \in [0, +\infty)$, who live in continuous time, and discount the future at a common rate $r > 0$. At each time t, agent t invests the fraction $k_t \in [0, 1]$ of her unit resource endowment in a *costly* and *risky* technology that produces *lagged* outcomes. Investing k_t immediately costs αk_t, with $\alpha > 0$, but is the only way to achieve an innovation and obtain a positive payoff. The innovation is achieved at the first time investment yields a *success*. In that case, the agent who made the successful investment receives a lump sum of 1 and the innovation process stops. Investment outcomes occur with an *outcome lag* $\Delta > 0$; thus, if the technology is good, investing k_t yields a *success* at time $t + \Delta$ at the first jump of a time-inhomogeneous Poisson process with instantaneous rate λk_t, where $\lambda > \alpha$. Therefore, if a success arrives at time t, agents who invested between $t - \Delta$ and t will not receive any payoff, even if their investment is successful. The technology is risky because it can also be bad, in which case investing in it will never produce a success.

An *investment profile* is a function $\mathbf{k} : \mathbb{R}_+ \to [0, 1]$. In order to guarantee that agents can always continuously update their beliefs based on the observation of past actions and outcomes, we restrict the analysis to *admissible* investment profiles, defined as profiles $\mathbf{k}$ such that $\int_{\underline{t}}^{\bar{t}} k_t dt$ is well defined for every $\underline{t} \leq \bar{t}$.

Agents observe the whole history of actions and outcomes but do not know whether the technology is good unless a success has arrived. The public belief that the technology is good conditional on no success having arrived at time t is denoted p_t, with $p_0 \in (0, 1)$ the common prior belief. As an agent's investment operates on the Poisson process with lag Δ, the ex-ante probability of a success arriving before time t is $p_0(1 - e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds})$. Therefore, the public belief at time t is:

$$p_t = \frac{p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds}}{1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds}}, \tag{1}$$

and follows the law of motion:

$$\dot{p}_t = -p_t(1 - p_t)\lambda k_{t-\Delta}\mathbb{1}_{t \geq \Delta}. \tag{2}$$

As the investment process stops after the first success, investing at time t may yield a positive payoff only if the technology is good and no success arrives between t and $t + \Delta$. The probability at time t that both conditions are satisfied is given by:[8]

$$\mu_t := p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t \geq \Delta}}^{t} k_s ds}, \tag{3}$$

which we refer to as the *attractiveness of investment* for agent t, reflecting its positive dependence on the agent's optimism and negative sensitivity to the risk of preemption.

Finally, when the integral $\int_0^{+\infty} k_t dt$ is well-defined, we refer to it as the *total investment* given profile $\mathbf{k}$. By integrating (2) over the interval $[0, +\infty)$, we establish a direct relationship

[8] Let $\bar{B}_t$ denote the event "no success arrives before time t", and G the event "the technology is good". If agent t has the opportunity to invest, $\bar{B}_t$ is true. Hence $\mu_t = P(\bar{B}_{t+\Delta} \cap G \mid \bar{B}_t) = P(\bar{B}_{t+\Delta} \mid G \cap \bar{B}_t)P(G \mid \bar{B}_t) = e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t \geq \Delta}}^{t} k_s ds} p_t$.

between the total investment and the asymptotic value of the common belief, denoted as $p_\infty(\mathbf{k})$:

$$\int_0^{+\infty} k_t dt = \frac{1}{\lambda} \ln\left(\frac{\Omega(p_\infty(\mathbf{k}))}{\Omega(p_0)}\right), \tag{4}$$

where $\Omega(p) := \frac{1-p}{p}$.

To fix ideas, let us consider a single player who must decide how much to invest in the technology. Investing an amount k entails an immediate cost of αk and yields a reward of 1 with a lag of Δ, provided that a success arrives, which occurs at rate λk if the technology is good. The single player's expected payoff is therefore given by:

$$v(k) := -\alpha k + e^{-r\Delta}\lambda k p_0.$$

It follows immediately that the single player will optimally invest the maximum amount ($k = 1$) if her initial optimism about the technology is sufficiently high, i.e., if $p_0 > \frac{\alpha}{\lambda}e^{r\Delta}$. Conversely, if this condition is not met, the single player chooses not to invest at all. The threshold value,

$$\underline{p} := \frac{\alpha}{\lambda}e^{r\Delta},$$

is referred to as the *single-player cutoff*, and represents the critical level of optimism required for a single player to invest.

3 The funding agency problem

We begin by examining the problem faced by a funding agency that coordinates and subsidizes individual investments while receiving a payoff of 1 upon the first success. Given an investment profile $\mathbf{k}$, if a success occurs at time $t < \infty$, the funding agency's payoff is:

$$e^{-rt} - \int_0^t e^{-rs}\alpha k_s ds,$$

while it is $-\int_0^\infty e^{-rs}\alpha k_s ds$ if a success never arrives. The probability that no success arrives before t is 1 if the technology is bad, and $e^{-\lambda\int_0^{(t-\Delta)\geq} k_s ds}$ if the technology is good. The expected payoff to the funding agency is thus:[9]

$$W(\mathbf{k}) = \int_0^\infty e^{-rt} \underbrace{(1 - p_0 + p_0 e^{-\lambda\int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds})}_{\text{Probability that no success arrives before } t} (\underbrace{-\alpha k_t + \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} p_t}_{\text{Expected instantaneous reward at time } t})dt. \tag{5}$$

The funding agency's problem is an optimal control problem with delay in the state variable p_t, which makes it difficult to solve in full generality. In the Appendix, we prove the existence of

[9] See Lemma 2 in Section A for the detailed calculations.

an optimal investment policy $\hat{\mathbf{k}}$ and use Pontryagin's principle to characterize its key properties. Here, we adopt a heuristic approach to clarify the trade-offs faced by the agency and to provide intuition for the structure of the optimal policy.

If the funding agency were myopic, it would choose k_t by trading off the current cost of investment against the discounted payoff $e^{-r\Delta}$ obtained in case of success at $t+\Delta$, provided no success occurs beforehand. The *net myopic marginal benefit* of k_t would therefore be

$$-\alpha + \lambda \mu_t e^{-r\Delta}.$$

A forward-looking agency, however, also internalizes that, in the absence of success, current investment affects the continuation value at $t+\Delta$ by increasing the duplication risk and lowering beliefs. Hence, the agency evaluates investment in terms of the discounted sum of all future myopic benefits it induces. This reasoning yields two implications.

First, the agency never invests if $p_0 \leq \underline{p}$. At $t = 0$, duplication risk is zero, so the myopic marginal benefit of investment is

$$-\alpha + \lambda p_0 e^{-r\Delta},$$

which is non-positive when $p_0 \leq \underline{p}$. Since beliefs can only decrease over time and future duplication risks are weakly higher than at $t = 0$, all future myopic marginal benefits are also non-positive. Investment is therefore never optimal.

Second, as long as $p_t > \underline{p}$, investment cannot stop permanently, which implies that beliefs converge to $\underline{p}$. Suppose instead that investment stops at some time τ while $p_\tau > \underline{p}$. Because duplication risk is zero at $\tau + \Delta$, a one-shot investment at $\tau + \Delta$ would yield the net marginal benefit

$$-\alpha + \lambda e^{-r\Delta} p_{\tau+\Delta}.$$

If $p_{\tau+\Delta}$ were strictly larger than $\underline{p}$, this expression would be positive and the agency would be better off resuming investment at $\tau + \Delta$. Hence, stopping investment at τ can be optimal only if $p_{\tau+\Delta} \leq \underline{p}$. Yet in that case, $p_{\tau+\Delta} < p_\tau$, which implies that some investment was undertaken between $\tau - \Delta$ and τ. This in turn implies that the attractiveness of investment at τ, namely μ_τ, is strictly smaller than $p_{\tau+\Delta}$, and therefore strictly smaller than $\underline{p}$.[10] The marginal benefit generated by the last investment at τ,

$$-\alpha + \lambda \mu_\tau e^{-r\Delta},$$

would therefore be negative, so the agency would have strictly preferred to stop investing before τ, a contradiction.

[10]By (1) and (3), $\mu_t = p_{t+\Delta}(1 - p_0 + p_0 e^{-\lambda \int_0^t k_s ds})/(1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds})$. Hence $\mu_t \leq p_{t+\Delta}$, with strict inequality if $\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t k_s ds > 0$.

Building on these observations and considering that the total amount of investment depends on the limit belief by (4), we can state the following proposition:

Proposition 1. *For every $\Delta \geq 0$, the funding agency's problem has a solution $\hat{\mathbf{k}}$. The total investment in $\hat{\mathbf{k}}$ is*

$$\hat{\tau}(p_0) := \frac{1}{\lambda} \max \left\{ 0, \ln \left(\frac{\Omega(\underline{p})}{\Omega(p_0)} \right) \right\}.$$

Proof. See Section A.2 in the Appendix. □

The remaining question is how optimally to allocate total investment $\hat{\tau}(p_0)$ over time when $p_0 > \underline{p}$. At each time t, the funding agency trades off the risk of wasteful duplication,

$$\rho_t := p_0(1 - e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}),$$

against the benefit of accelerating success. Because ρ_t is negligible at the outset, full investment ($\hat{k}_t = 1$) is initially optimal. By Proposition 1, however, this phase of full investment must end before $\hat{\tau}(p_0)$. We are thus left with two classes of candidate policies: (*i*) *decreasing policies*, with investment decreasing over time; and (*ii*) *non-monotonic policies*, with investment fluctuating over time.

When there is no outcome lag, duplication never arises ($\rho_t = 0$). Thus, the optimal policy is the *cutoff policy*: $\hat{k}_t = 1$ up to $\hat{\tau}(p_0)$ and 0 thereafter.[11] When $\Delta > 0$, in contrast, optimal policies cannot be decreasing. Suppose, for contradiction, that there exists a decreasing optimal policy $\hat{k}_t$. Such a policy must either (a) reach a finite time after which investment ceases permanently, or (b) eventually enter an everlasting period of decreasing, interior, investment $\hat{k}_t \in (0, 1)$. Consider case (a), and let $\hat{\tau}$ denote the earliest time after which the agency stops investing forever. Because duplication risk depends on recent investment, the agency could strictly improve upon this policy by slightly reducing investment just before $\hat{\tau}$, thereby lowering duplication risk at $\hat{\tau}$, and resuming investment Δ periods later if no success occurs. This argument, formalized in Lemma 6 in the Appendix, implies that permanent cessation of investment cannot be optimal. Next, consider case (b). An interior investment path requires indifference at every $t \geq \hat{\tau}$. We prove in Section A.3 of the Appendix that the indifference condition at time t relates future optimal investment to current duplication risk as follows:

$$\hat{k}_{t+\Delta} = e^{r\Delta} \frac{p_0}{p_0 - \rho_t} \left(\hat{k}_{t-\Delta} \mathbb{1}_{t\geq\Delta} + \frac{r}{\lambda p_t} \right) - \frac{r}{\alpha} \text{ for all } t \geq \hat{\tau}. \tag{6}$$

[11] When $\Delta = 0$, the funding agency's objective (5) simplifies to $W(\mathbf{k}) = \int_0^\infty e^{-rt}(1 - p_0 + p_0 e^{-\lambda \int_0^t k_s ds}) k_t(-\alpha + \lambda p_t)dt$. As long as $p_t > \alpha/\lambda$, the integrand is positive, making full investment optimal. Since p_t strictly decreases when $\hat{k}_t = 1$, it reaches $\underline{p}$ in finite time τ, after which it is optimal to stop investment.

The indifference condition has a simple interpretation. Reducing investment at time t lowers duplication risk but also reduces the probability of success at $t+\Delta$. To maintain indifference, this lower short-run investment must be offset by higher investment at $t+\Delta$. Thus, $\hat{k}_{t+\Delta}$ decreases with investment at t, and therefore increases with duplication risk ρ_t. This observation rules out monotonicity. If full investment were to stop before the first outcomes are realized (i.e., $\hat{\tau} < \Delta$), duplication risk would strictly increase on $[\hat{\tau}, \Delta)$, implying through (6) that investment would also increase on $[\hat{\tau}+\Delta, 2\Delta)$, a contradiction. Conversely, if $\hat{\tau} > \Delta$, duplication risk would attain its maximal value at $\hat{\tau}$, since investment equals 1 on $[\hat{\tau}-\Delta, \hat{\tau}]$. We prove in section A.3 that the compensating level $k_{\hat{\tau}+\Delta}$ required by (6) would then exceed 1, which is infeasible.

Therefore, with an outcome lag, an optimal investment path cannot be monotone. Instead, it must be non-monotonic, exhibiting what we refer to as *investment waves*. The mechanism is as follows. Because pessimism increases over time while the marginal cost of investment remains constant, the agency initially reduces investment. Yet lower investment decreases future duplication risk, eventually restoring incentives to experiment despite heightened pessimism. Investment waves thus arise from the dynamic interaction between growing pessimism and the endogenous evolution of duplication risk.

Proposition 2 (Investment waves)**.** *Let $p_0 > \underline{p}$. In the agency problem, every optimal policy $\hat{k}$ is non-monotonic if and only if the outcome lag is positive.*

Proof. See Section A.3 in the Appendix. □

We now turn to the analysis of the strategic problem and compare the equilibrium with the agency's optimal solution.

4 The strategic problem

In this section, we analyze the problem where each player determines her investment by best-responding to past investments. Since player t invests only at time t, an action for player t is $k_t \in [0,1]$, and an action profile is a function $\mathbf{k} : \mathbb{R}_+ \to [0,1]$. As before, we restrict our analysis to admissible action profiles, ensuring that $\int_{\underline{t}}^{\bar{t}} k_t dt$ is well-defined for every $\underline{t} \leq \bar{t}$.

At time t, it is common knowledge that no success has occurred yet, thus that all investments made by players in the interval $[0, (t-\Delta)\mathbb{1}_{t\geq\Delta})$ were unsuccessful. However, player t recognizes the possibility of being preempted by one of the players in $[(t-\Delta)\mathbb{1}_{t\geq\Delta}, t)$. Thus, investing k_t will yield a payoff of 1 at time $t+\Delta$ if and only if the following conditions are met: 1) no success arrives between t and $t+\Delta$, 2) the technology is good and 3) player t's investment is successful. Moreover, regardless of the outcome, the investment immediately incurs a cost of αk_t. Letting $\mathbf{k}_{-t}$ denote the investment profile of all players in $[0, t)$, the expected payoff for player t is given by:

$$u(k_t; \mathbf{k}_{-t}) := \underbrace{-\alpha k_t}_{\text{instantaneous cost}} + \underbrace{e^{-r\Delta}\lambda k_t \mu_t}_{\text{expected benefit}}, \tag{7}$$

which can conveniently be rewritten $u(k_t, \mathbf{k}_{-t}) = \lambda e^{-r\Delta} k_t(\mu_t - \underline{p})$. The linearity in k_t in the payoff expression implies that the best response of player t to $\mathbf{k}_{-t}$ depends on the attractiveness of investment as follows:

$$k_t \begin{cases} = 1 & \text{if } \mu_t > \underline{p}, \\ \in [0,1] & \text{if } \mu_t = \underline{p}, \\ = 0 & \text{if } \mu_t < \underline{p}. \end{cases} \tag{8}$$

In the absence of an outcome lag ($\Delta = 0$), the risk of preemption is nonexistent, implying that $\mu_t = p_t$ for all t. Consequently, investment attractiveness declines over time, because of players' increasing pessimism about the technology. This no longer holds when $\Delta > 0$. With an outcome lag, investment attractiveness also depends on the intensity of competition faced by the player. As a result, while $\mu_0 = p_0$, we have $\mu_t \leq p_t$ for all $t > 0$, reflecting the added complexity introduced by potential preemption. Differentiating μ_t with respect to t and using (2), we obtain:

$$\dot{\mu}_t = -\mu_t \lambda(k_t - p_t k_{t-\Delta} \mathbb{1}_{t \geq \Delta}). \tag{9}$$

It is immediately evident that the growth rate of investment attractiveness decreases with k_t and increases with $k_{t-\Delta}$. The first effect is straightforward: When player t invests more, her opponents face a stronger competition, which can only reduce the attractiveness of investment. The second effect, however, is less intuitive. To understand why, compare the situations faced by player t and her "immediate" successor, player $t + dt$. First, the two players face different preemption risks, because, unlike player t, player $t + dt$ competes with player t but not with player $t - \Delta$. As a result, the risk of preemption changes by $\lambda(-k_{t-\Delta}\mathbb{1}_{t\geq\Delta} + k_t)$. Second, player $t + dt$ is more pessimistic than player t because, unlike player t, she knows that player $t - \Delta$ did not obtain a success. This increases the subjective probability of facing a bad technology by $\lambda(1 - p_t)k_{t-\Delta}\mathbb{1}_{t\geq\Delta}$. Yet, higher investment by player $t - \Delta$ reduces the risk of preemption and amplifies pessimism, which becomes clear when we rearrange (9) as follows:

$$\frac{\dot{\mu}_t}{\mu_t} = \underbrace{\lambda(k_{t-\Delta}\mathbb{1}_{t\geq\Delta} - k_t)}_{\text{Effect on the risk of preemption}} \underbrace{-\lambda(1 - p_t)k_{t-\Delta}\mathbb{1}_{t\geq\Delta}}_{\text{Effect on optimism}}.$$

The investment of player $t - \Delta$ thus has opposite effects on the variation of μ_t: A positive effect, via the risk of preemption, and a negative effect, via the probability that innovation is impossible. The overall impact of $k_{t-\Delta}$ on $\dot{\mu}_t$ turns out to be positive. In other words, past investment enhances the attractiveness of investment over time, primarily because the reduction in preemption risk outweighs the growing pessimism.

Several observations follow. If $k_{t-\Delta}\mathbb{1}_{t\geq\Delta} = 0$ or $k_t = 1$, then μ_t is necessarily non-increasing because the negative impact of current investment dominates. If $k_t = 0$, the only effect at play is the positive impact of $k_{t-\Delta}$, making the attractiveness of investment non-decreasing. When both $k_{t-\Delta}\mathbb{1}_{t\geq\Delta}$ and k_t are positive, the sign of $\dot{\mu}_t$ will be endogenously determined in equilibrium.

4.1 Equilibrium characterization

Since player 0 faces no competition and is the most optimistic among all players, she has the strongest incentives to invest. Consequently, if player 0 chooses not to invest, no resources will ever be invested in the technology. Given that $\mu_0 = p_0$, player 0 follows a simple decision rule: She invests if and only if she is sufficiently optimistic, i.e., $p_0 > \underline{p}$. This explains why no investment occurs in equilibrium when $p_0 \leq \underline{p}$. Therefore, the remainder of the analysis focuses on the case where $p_0 > \underline{p}$.

If $p_0 > \underline{p}$, player 0 is confident enough in the technology to invest fully ($k_0 = 1$). Her immediate successors, facing minimal competition and holding the same belief as player 0, also choose to fully invest. However, as long as players continue investing at this rate, the attractiveness of investment declines according to (9). Eventually, it reaches the threshold $\underline{p}$ at a finite cutoff time τ, at which

$$\mu_\tau = \underline{p}.$$

Since $k_t = 1$ for all $t < \tau$, the attractiveness of investment for player τ is given by $\mu_\tau = p_0 e^{-\lambda\tau}/(1 - p_0 + p_0 e^{-\lambda(\tau-\Delta)\mathbb{1}_{\tau\geq\Delta}})$. The value of τ depends on whether $\tau < \Delta$ or $\tau \geq \Delta$, which in turn depends on p_0. Now, what happens to investment after player τ? For intuition, let $\tau + dt$ denote an "immediate" successor of player τ. If $\mu_{\tau+dt} < \underline{p}$, player $\tau + dt$ will not invest, according to (8). However, by (9), this would cause μ_t to weakly increase at τ , leading to $\mu_{\tau+dt} \geq \underline{p}$, which contradicts our assumption. Therefore, $\mu_{\tau+dt}$ must be at least $\underline{p}$. If $\mu_{\tau+dt}$ were strictly greater than $\underline{p}$, player $\tau + dt$ would invest 1 by (8). However, this would lead the attractiveness of investment to weakly decrease in $\tau + dt$ by (9), contradicting $\mu_{\tau+dt} > \underline{p}$. Thus, the attractiveness of investment must also equal $\underline{p}$ for the immediate successor of player τ, as well as for all future players:

$$\mu_t = \underline{p} \; \forall \; t \geq \tau.$$

Since all players $t \geq \tau$ are indifferent about whether to invest, the attractiveness of investment remains constant beyond τ, meaning $\dot{\mu}_t = 0$ for every $t > \tau$. Substituting this condition into (9) yields the equilibrium action of almost all players after τ:

$$k_t = p_t k_{t-\Delta} \mathbb{1}_{t \geq \Delta} \; \forall \; t \geq \tau.$$

Notably, in equilibrium, almost every player t arriving after τ best responds to the investment made by their predecessor $t-\Delta$ by investing a smaller amount: $k_t \leq k_{t-\Delta}$. A crucial consequence is that if player $t - \Delta$ did not invest, then player t also refrains from investing.

Since k_0 is uniquely determined by p_0, and each player t best responds based on the sequence of actions taken by all players in $[0, (t-\Delta)\mathbb{1}_{t \geq \Delta})$, the equilibrium strategy profile $\mathbf{k}^*$ is *essentially unique*.[12] The following proposition formally characterizes this equilibrium.

Proposition 3 (Equilibrium characterization)**.** *Let* $p_0 > \underline{p}$. *There is an essentially unique equilibrium* $\mathbf{k}^*$*, such that:*

$$k_t^* = \begin{cases} 1, & \textit{if}\, t < \tau^*(p_0), \\ p_t k_{t-\Delta}^* \mathbb{1}_{t \geq \Delta}, & \textit{if}\, t \geq \tau^*(p_0), \end{cases}$$

$$\textit{where } \tau^*(p_0) := \min\{t \mid \mu_t = \underline{p}\} = \begin{cases} \frac{1}{\lambda} \ln\left(\frac{p_0}{\underline{p}}\right), & \textit{if}\, p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta}], \\ \Delta + \frac{1}{\lambda} \ln\left(\frac{\Omega(\underline{p}e^{\lambda\Delta})}{\Omega(p_0)}\right), & \textit{if}\, p_0 \geq \underline{p}e^{\lambda\Delta}. \end{cases}$$

Proof. See Section B.1 in the Appendix. □

4.2 Investment breaks

In equilibrium, the investment of almost all agents t who play after $\tau^*(p_0)$ follows $k_t^* = p_t k_{t-\Delta}^* \mathbb{1}_{t \geq \Delta}$. Rearranging this equation, we obtain:

$$k_t^* = (\prod_{m=0}^{n_t} p_{t-m\Delta}) \mathbb{1}_{t \geq (n_t+1)\Delta}, \tag{10}$$

where n_t is the unique integer such that $t \in [\tau^*(p_0) + n_t\Delta, \tau^*(p_0) + (n_t + 1)\Delta)$.

When $\tau^*(p_0) < \Delta$, $\tau^*(p_0) + n_t\Delta < (n_t + 1)\Delta$ for each t. Therefore, the Dirac function in (10) is equal to 0 for all players $t \in [\tau^*(p_0) + n_t\Delta, (n_t + 1)\Delta)$. The intuition behind this

[12] By essential uniqueness, we mean that equilibrium actions are unique up to the actions of any null set of indifferent players, which are arbitrary.

as follows: Players in the interval $[\tau^*(p_0), \Delta)$ have not yet received feedback from the first investments, but are aware that their predecessors invested the maximum possible resource into the technology. Given the high risk of preemption, they opt not to invest at all. Since, in equilibrium, choosing not to invest is a best response for player t when player $t-\Delta$ has also refrained from investing, the start of this initial *investment break* replicates periodically at every point in time $s \in \{\tau^*(p_0) + n\Delta,\ n \in \mathbb{N}\}$. Conversely, when $\tau^*(p_0) > \Delta$, the Dirac function in (10) remains equal to 1 for all t, meaning no investment break occurs. This pattern is illustrated in Figure 1.

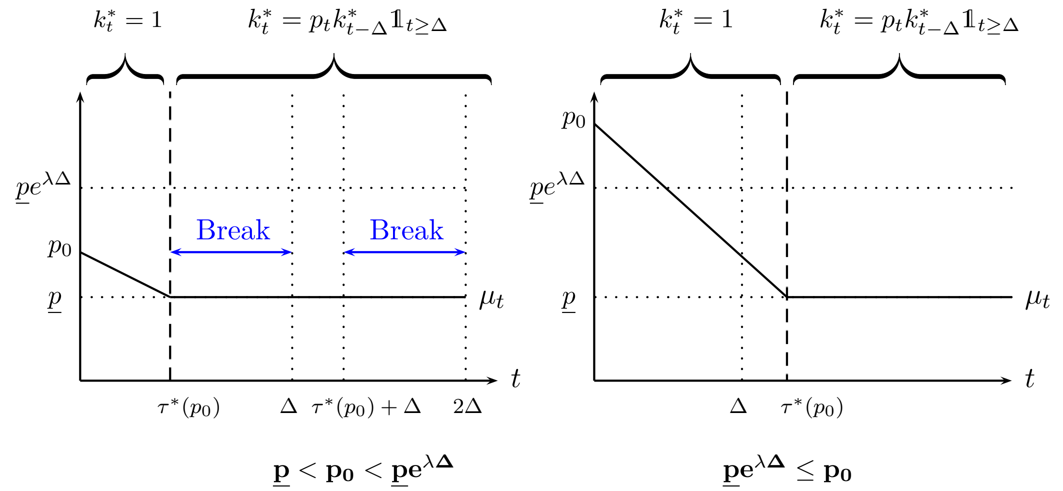


Figure 1: Equilibrium dynamics of μ_t when $p_0 > \underline{p}$.

Regardless of the value of Δ, the initial investment period $[0, \tau^*(p_0))$, during which players invest all of their resources, induces subsequent periods of positive investment starting at every time $s \in \{n\Delta,\ n \in \mathbb{N}\}$. In the Appendix, we solve equation (10) and prove that, when it is positive, the equilibrium investment takes the form $k_t^* = \phi(n,t)$ where $\phi(n,t)$ is defined for every $(n,t) \in \mathbb{N} \times \mathbb{R}_+$ by

$$\Omega(\phi(n,t)) := \frac{\Omega(\underline{p}^n)\Omega(p_0)}{\Omega(\underline{p})} e^{\lambda(t-n\Delta)}.$$

The following proposition summarizes these findings, encapsulating the structure of equilibrium investment dynamics and the periodic nature of investment behavior over time.

Proposition 4 (Investment breaks)**.**

- *If* $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta}]$, $k_t^* = 1$ *if* $t \in [0, \tau^*(p_0))$, *and,* $\forall\, n \in \mathbb{N}^*$,

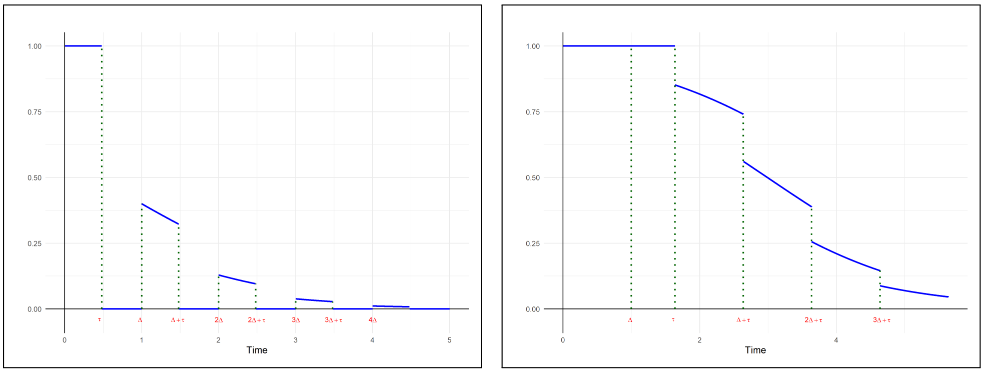
$$k_t^* = \begin{cases} 0 & \text{if } t \in [\tau^*(p_0) + (n-1)\Delta, n\Delta), \\ \phi(n,t) & \text{if } t \in [n\Delta, \tau^*(p_0) + n\Delta). \end{cases}$$

- *If* $p_0 > \underline{p}e^{\lambda\Delta}$, $k_t^* = 1$ *if* $t \in [0, \tau^*(p_0))$ *and,* $\forall\, n \in \mathbb{N}^*$,

$$k_t^* = \phi(n,t) \text{ if } t \in [\tau^*(p_0) + (n-1)\Delta, \tau^*(p_0) + n\Delta).$$

Proof. See Section B.2 in the Appendix. □

The equilibrium investment dynamics are illustrated in the figures below for the parameter values $\alpha = 0.2$, $\lambda = 0.7$, $r = 0.001$ and $\Delta = 1$, which imply $\underline{p} \approx 0.29$ and $\underline{p}e^{\lambda\Delta} \approx 0.58$. Figure 2a corresponds to the case $p_0 = 0.4$, while Figure 2b considers $p_0 = 0.9$.



(a) $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta})$

(b) $p_0 > \underline{p}e^{\lambda\Delta}$

Figure 2: Equilibrium investment dynamics.

Several observations follow. First, the likelihood of investment breaks in a risky technology increases as the outcome lag grows. This is because the lag creates de facto competition between each player t and all players in $[t-\Delta, t)$. The larger Δ, the more intense the competition. When Δ is relatively small compared to players' optimism ($\underline{p}e^{\lambda\Delta} < p_0$), competition does not dissuade any player from investing. In this case, all players invest a positive amount, gradually decreasing their investment over time as they become more pessimistic about the technology, even though the risk of preemption also declines. In contrast, when Δ is large ($p_0 < \underline{p}e^{\lambda\Delta}$), some players opt not to invest as the risk of being preempted by past competitors is too high. As the period of non-investment extends, the risk of preemption gradually declines until investment becomes attractive again. This results in a non-monotonic risk of preemption: For any player choosing not to invest, there exists a more pessimistic player who eventually does.

Second, in equilibrium, the proportion of time during which players do not invest is

$$f^*(p_0) := \max\left\{0, 1 - \frac{\tau^*(p_0)}{\Delta}\right\},$$

which can be interpreted as the *frequency* of investment breaks. It follows directly that the frequency of investment breaks increases with Δ and α but decreases with p_0. The intuition is as follows. When Δ increases, the discounted reward in the event of success mechanically decreases, causing $\underline{p}$ to increase. As a result, the period during which investment remains attractive (i.e., above $\underline{p}$) shortens, leading to a decrease in $\tau^*(p_0)$. Consequently, the ratio $\tau^*(p_0)/\Delta$ declines, making investment breaks more frequent. Similarly, an increase in investment cost increases $\underline{p}$, further reducing $\tau^*(p_0)$. Finally, the lower p_0, the faster investment attractiveness declines to $\underline{p}$, shortening $\tau^*(p_0)$ and increasing the frequency of investment breaks. The effect of investment productivity λ on the frequency of investment breaks is more nuanced, as higher λ has opposite effects on $\tau^*(p_0)$. On the one hand, a higher λ increases the risk of preemption, reducing the attractiveness of investment and causing μ_t to reach $\underline{p}$ sooner. On the other hand, a higher λ also improves the probability of success when the technology is good, which lowers $\underline{p}$ and delays the point at which μ_t reaches $\underline{p}$. Differentiating $\tau^*(p_0)$ with respect to λ, we obtain:

$$\frac{\partial \tau^*(p_0)}{\partial \lambda} = \underbrace{-\frac{1}{\lambda^2} \ln\left(\frac{p_0}{\underline{p}}\right)}_{\text{Effect via risk of preemption}} + \underbrace{\frac{1}{\lambda^2}}_{\text{Effect via success probability}}$$

The negative effect is weighted by the prior belief p_0, as preemption occurs only if the technology is good. In contrast, the positive effect is independent of p_0 since it stems from the increased probability of success *conditional* on the technology being good. Consequently, the positive effect dominates for small values of p_0 (i.e., $\ln(p_0/\underline{p}) < 1$), while the negative effect prevails otherwise.

4.3 Inefficiency

A natural question is whether the equilibrium investment profile is efficient in maximizing the funding agency's objective. When there is no outcome lag, the answer is straightforward: The equilibrium strategy aligns perfectly with the agency's optimal policy.[13] With an outcome lag, however, the answer is less obvious because the only established properties of the optimal policy are that it is non-monotonic and that the total investment over time is $\hat{\tau}(p_0)$. Yet, this is enough to prove the inefficiency of equilibrium.

To do so, we observe that the funding agency's payoff can be rewritten as the discounted sum of individual payoffs. From this observation, it follows almost immediately that the equilibrium is inefficient for all $\Delta > 0$. Indeed, in equilibrium, players invest 1 up to $\tau^*(p_0)$ and receive payoff 0 thereafter. Consequently, the funding agency's payoff in equilibrium is identical to the payoff of a cutoff policy in which $k_t = 1$ if $t < \tau^*(p_0)$ and $k_t = 0$ otherwise. However, we formally prove (see Lemma 6 in the Appendix) that a cutoff policy cannot be optimal for the funding agency. The underlying intuition is that the agency can always improve upon any

[13]When $\Delta = 0$, we have $k_t^* = \hat{k}_t = \mathbb{1}_{t<\hat{\tau}(p_0)}$.

cutoff policy by reallocating some of the resources invested at the cutoff point to a slightly later time, provided that no success has yet been achieved. This proves what follows:

Proposition 5. *Let $p_0 > \underline{p}$. The equilibrium $\mathbf{k}^*$ is inefficient if and only if $\Delta > 0$.*

Proof. See Section B.3 in the Appendix □

What causes the inefficiency? In most experimentation games, players tend to free-ride on each other's investments, which leads to an underprovision of investment relative to various benchmarks, in particular the cooperative solution. However, in our model, each agent invests only once, which eliminates free-riding in the traditional sense. In fact, the total investment in equilibrium is identical to that in the funding-agency solution, i.e.,

$$\int_0^{+\infty} k_t^* dt = \hat{\tau}(p_0).$$

This result directly follows from the fact that the public belief in equilibrium converges to the single-player cutoff $\underline{p}$. Indeed, no player is willing to invest when the belief falls below $\underline{p}$, as the expected return on investment is negative even in the absence of competition. Conversely, if a player were facing no competition with a common belief larger than $\underline{p}$, she would have strict incentives to invest. This implies that investment does not stop at belief levels above $\underline{p}$.

Therefore, iefficiency arises from a misallocation over time of the total investment level $\hat{\tau}(p_0)$. The distortion stems from the fact that, unlike the funding agency, player t has only a one-shot opportunity to invest and therefore has a higher cost of reducing investment. By contrast, the agency can reallocate effort over time and wait for the outcomes of past investments before committing further resources. Because players cannot smooth investment intertemporally, equilibrium places too much weight on early investment relative to the agency's optimum. This divergence is particularly striking when the outcome lag is small: for $p_0 > \underline{p}e^{\lambda\Delta}$, equilibrium investment is strictly decreasing, whereas the funding-agency solution exhibits non-monotonic dynamics.

5 Extensions

In the previous section, we demonstrated that, for sufficiently large outcome lags, equilibrium dynamics become non-monotonic, featuring infinitely many investment breaks. In this section, we analyze three extensions of our model and prove that non-monotonicities persist with uncertain outcome lags, convex investment costs, and endogenous timing.

5.1 Uncertain outcome lag

In our baseline model, the outcome lag is deterministic and common knowledge among all players. This assumption is reasonable in contexts where the investment technology is risky but the outcome lag is predictable (e.g., in agriculture, farmers know that a seed planted in March will either sprout in June if viable or never germinate at all). However, in many settings, the outcome lag itself is uncertain. To account for this, we extend our model to consider a scenario where the outcome lag is unknown to players. Specifically, players are assumed to know that the outcome lag is a random variable Δ with a probability density function f over a support $[\underline{\Delta}, \overline{\Delta}]$. In the absence of success, players update their beliefs about both the realization of Δ and the technology's type. We show that, as in the deterministic case, equilibrium exhibits investment breaks if the expected outcome lag is sufficiently large in a sense that will be made clear later on. However, in contrast to the deterministic setting, the number of investment breaks remains finite.

Conditional on no success having arrived before t, the expected payoff to player t is

$$u(k_t; k_{-t}) = -\alpha k_t + \mathrm{E}[e^{-r\Delta}]\lambda k_t \frac{p_0 e^{-\lambda \int_0^t k_s ds}}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds}]},$$

where E is the unconditional expectation.[14]

As in the main model, the linearity of this expression in k_t implies that the best response of player t to k_{-t} is as follows:

$$k_t \begin{cases} = 1 & \text{if } \xi_t > \underline{p}, \\ \in [0,1] & \text{if } \xi_t = \underline{p}, \\ = 0 & \text{if } \xi_t < \underline{p}, \end{cases}$$

where the cutoff is now defined by $\underline{p} := \frac{\alpha}{\lambda \mathrm{E}[e^{-r\Delta}]}$ and the attractiveness of investment by

$$\xi_t := \frac{p_0 e^{-\lambda \int_0^t k_s ds}}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t \geq \Delta}} k_s ds}]}.$$

The mechanics of the equilibrium construction are very similar to the case where Δ is known. Because of the risk of preemption, the attractiveness of investment is always lower than the common belief, meaning no player invests in equilibrium when $p_0 \leq \underline{p}$. Therefore, the remainder of the analysis focuses on the case where $p_0 > \underline{p}$. In the Appendix, we prove that $\dot{\xi}_t < \lambda \xi_t (1 - k_t)$, implying that investment becomes less attractive when players are fully investing. Given this and the initial condition $\xi_0 = p_0$, it follows that when $p_0 > \underline{p}$, there exists

[14] See Section C in the Appendix for detailed calculations.

a cutoff $\tau_u^*(p_0) > 0$ such that $\xi_t > \underline{p}$ for all $t \in [0, \tau_u^*(p_0))$, and $\xi_{\tau_u^*(p_0)} = \underline{p}$. As in the main model, the attractiveness of investment ξ_t remains equal to $\underline{p}$ indefinitely after $\tau_u^*(p_0)$, and the best response of players arriving after this point is a.e. given by the indifference condition $\dot{\xi}_t = 0$ for all $t \geq \tau_u^*(p_0)$.

Proposition 6 (Equilibrium characterization)**.** *Let $p_0 \in (\underline{p}, 1)$. There is an essentially unique equilibrium $\mathbf{k}^*$, such that:*

$$k_t^* = \begin{cases} 1 & \textit{if } t < \tau_u^*(p_0), \\ \frac{p_0 \mathrm{E}[k_{t-\Delta}^* \mathbb{1}_{t \geq \Delta} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s^* ds}]}{1-p_0+p_0\mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s^* ds}]} & \textit{if } t \geq \tau_u^*(p_0), \end{cases} \tag{11}$$

where $\tau_u^*(p_0) := \min\{t \mid \xi_t = \underline{p}\}$.

Proof. See Section C in the Appendix. □

In the certain setting, when players *know* that Δ is large enough, there are infinitely many equilibrium breaks. A similar pattern emerges in the uncertain case, albeit with some noticeable differences. Under some conditions on the distribution of Δ, the attractiveness of investment reaches $\underline{p}$ before $\underline{\Delta}$, thus at a moment when players are certain that no outcome has been realized yet. In this case, players pause investment between $\tau_u^*(p_0)$ and $\underline{\Delta}$, like they do in the certain case between $\tau^*(p_0)$ and Δ, as the high risk of preemption makes investment too risky. In the certain setting, the initial investment break leads to an infinite sequence of breaks of the same length. In the uncertain setting, the initial break induces a subsequent investment break only if its duration exceeds $\overline{\Delta} - \underline{\Delta}$. Each successive break that follows is shorter than the preceding one, triggering another pause only if it still exceeds $\overline{\Delta} - \underline{\Delta}$. As these breaks progressively shorten, the equilibrium eventually transitions into an infinite period of positive investment. Moreover, there is a unique jump discontinuity of k_t^* at $\tau_u^*(p_0)$. These key distinctions from the setting with deterministic Δ are formally outlined in the next proposition.

Proposition 7 (Investment breaks)**.** *If $\frac{\alpha}{\lambda \mathrm{E}[e^{-r\Delta}]} < p_0 < \frac{\alpha}{\lambda \mathrm{E}[e^{-r\Delta}]} e^{\lambda \underline{\Delta}}$, then $0 < \tau_u^*(p_0) < \underline{\Delta}$ and*

$$k_t^* \begin{cases} = 1 & \textit{if } t < \tau_u^*(p_0), \\ = 0 & \textit{if } t \in [\tau_u^*(p_0) + (n-1)\overline{\Delta}, n\underline{\Delta}) \textit{ for each integer } n \leq \bar{n}_u(p_0), \\ > 0 & \textit{otherwise,} \end{cases}$$

with $\bar{n}_u(p_0) := \max\{n \in \mathbb{N},\ n < \frac{\overline{\Delta} - \tau_u^*(p_0)}{\overline{\Delta} - \underline{\Delta}}\}$.

Proof. See Section C in the Appendix. □

As there are exactly $\bar{n}_u(p_0)$ investment breaks, there are at least $\bar{n}_u(p_0)$ investment waves in equilibrium.

5.2 Convex investment costs

In the baseline model, we assume linear investment costs. This is a reasonable assumption for industrial sectors where returns are constant but may be less appropriate for sectors that are characterized by decreasing returns. In this section, we extend our analysis to a setting with convex investment costs. Specifically, we assume that the cost of investing $k_t \in \mathbb{R}^+$ is given by $\alpha k_t^2/2$. In contrast to the linear setting, investment in this framework is continuous and always positive. Despite this difference, investment waves can still emerge in equilibrium, albeit under conditions that differ qualitatively from those in the linear case.

Conditional on no success having arrived before time t, the expected payoff to player t is:

$$u(k_t; k_{-t}) = -\frac{\alpha}{2}\, k_t^2 + e^{-r\Delta} \lambda k_t p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}. \tag{12}$$

As the latter expression is concave in k_t, the best response of player t to k_{-t} is:

$$k_t = \frac{\mu_t}{\underline{p}},$$

where the single player cutoff $\underline{p}$ and the attractiveness of investment μ_t are defined as in the main model. The next result immediately follows.

Proposition 8 (Equilibrium characterization)**.** *There is a unique equilibrium* $\mathbf{k}^*$*, such that:*

$$k_t^* = \frac{p_0}{\underline{p}} \frac{e^{-\lambda \int_0^t k_s^* ds}}{1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s^* ds}} \;\forall\, t.$$

The first difference with the linear setting is immediately obvious: Investment remains continuous and strictly positive at all times. This implies that investment breaks are unique to the linear case, suggesting that the lumpiness of investment dynamics depends on the nature of returns. However, investment waves can still emerge in equilibrium under convex costs, albeit under qualitatively different conditions. In the linear setting, waves occur when the outcome lag is sufficiently large, as the risk of preemption intensifies just before the first outcomes materialize. By contrast, in the convex setting, investment waves arise only when the outcome lag falls within an intermediate range. Moreover, unlike in the linear case, the number of investment waves is always finite.

The reason is as follows. The attractiveness of investment is differentiable everywhere but in Δ. Plugging $k_t = \mu_t/\underline{p}$ into the law of motion of μ_t, we obtain

$$\dot{\mu}_t = -\mu_t \frac{\lambda}{\underline{p}} (\mu_t - p_t \mu_{t-\Delta} \mathbb{1}_{t\geq\Delta}) \text{ for all } t \neq \Delta. \tag{13}$$

As in the linear setting, the attractiveness of investment decreases before Δ, leading players to invest progressively smaller amounts in the technology until time Δ. The subsequent investment dynamics, however, hinge on the behavior of player Δ. We prove in the Appendix that if investment continues to decline immediately after Δ, it will do so indefinitely. Conversely, if investment increases after Δ, then equilibrium dynamics will exhibit a finite number of periods where investment rises, creating investment waves. In other words, investment waves occur in equilibrium if and only if $\dot{\mu}_\Delta > 0$, which occurs precisely when the attractiveness of investment at time Δ is sufficiently low, precisely when

$$\mu_\Delta < p_0^2.$$

This condition is equivalent to $p_0 > 2/(1 + \sqrt{1 + 4\lambda^2/(\alpha re)})$ and $\Delta \in [\underline{\Delta}_c, \overline{\Delta}_c]$, where $\underline{\Delta}_c$ and $\overline{\Delta}_c$ are the solutions of $\Delta e^{-r\Delta} = \alpha(1-p_0)/(\lambda p_0)^2$. This result is formally stated in the following proposition.

Proposition 9 (Investment waves)**.**

- *If* $p_0 > \frac{2}{1+\sqrt{1+4\lambda^2/(\alpha re)}}$ *and* $\Delta \in [\underline{\Delta}_c, \overline{\Delta}_c]$*, there is a sequence of* $\bar{n}_c(p_0) < +\infty$ *intervals of* $[\Delta, +\infty)$ *such that* $\dot{k}_t^* > 0$ *for all* t *in the union of these intervals, and* $\dot{k}_t^* < 0$ *otherwise.*
- *Otherwise,* $\dot{k}_t^* < 0$ *for all* t*.*

Proof. See Section D in the Appendix. □

Why does an excessive lag prevent investment waves? This is because an increased Δ has opposite effects on the attractiveness of investment at time Δ. Integrating (13) from 0 to Δ yields:

$$\mu_\Delta = \frac{p_0 \underline{p}}{\underline{p} + \lambda p_0 \Delta}.$$

This equation reveals two opposing forces. First, there is a direct negative effect: Since μ_t decreases before Δ, a larger Δ leads to a smaller μ_Δ. The same effect operates in the linear setting. In the convex setting, there is also an indirect positive effect via the increase in $\underline{p}$. A longer lag raises $\underline{p}$, which reduces k_t for all $t < \Delta$, thereby making investment more attractive by lowering the risk of preemption. The net effect depends on Δ: μ_Δ decreases with Δ for $\Delta \in [0, 1/r]$ but increases for $\Delta \in [1/r, +\infty)$. Investment waves emerge only if μ_Δ remains sufficiently low, which occurs when $\Delta \in [\underline{\Delta}_c, \overline{\Delta}_c]$.

5.3 Endogenous investment timing

In the baseline model, investment timing is exogenous. This is a stylized way of representing sequential generations of investors, such as cohorts of PhD students who must choose a research topic. A natural extension is to allow players to decide not only *how much*, but also *when*, to invest. Specifically, each player $i \in \mathbb{R}_+$ chooses a pure strategy $\sigma_i = (t_i, k_i) \in \mathbb{R}_+ \times [0,1]$, with the interpretation that player i invests k_i at time t_i if no success has occurred before t_i. If multiple players invest at time t and no successes occur between t and $t+\Delta$, then the payoff of 1 is equally shared at time $t+\Delta$ among all successful investors. We prove that, as in the setting with exogenous timing, equilibrium investment breaks emerge if the outcome lag is sufficiently large.

We denote by $K_t := \int_0^\infty k_i \mathbb{1}_{\sigma_i=(t,k_i)} di$ the aggregate investment at time t given the strategy profile $(\sigma_i)_i$. Accordingly, $\mathbf{K} : \mathbb{R}_+ \to \mathbb{R}_+$ denotes the *investment distribution* over $\mathbb{R}_+$. Note that $K_t > 0$ if and only if a positive mass of players invest at time t. We restrict the analysis to *atomic equilibria*, defined as equilibrium strategy profiles such that no player invests at times when the aggregate investment is 0. Formally, $\sigma = (t_i, k_i)_i$ is atomic if, for all i, $k_i > 0$ only if $K_{t_i} > 0$.

Fix an atomic profile σ and the induced investment distribution $\mathbf{K}$. If a player invests at time t, obtains a success and is not preempted by a predecessor, she obtains in expectation:[15]

$$F(K_t) := \begin{cases} \frac{1-e^{-\lambda K_t}}{\lambda K_t} & \text{if } K_t > 0 \\ 1 & \text{if } K_t = 0. \end{cases}$$

The expected payoff yielded by strategy (t,k) to a given player is

$$u(t,k;\mathbf{K}) = P(\bar{B}_t) e^{-rt} \lambda e^{-r\Delta} k \left(-\underline{p} + F(K_t) p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t K_s ds} \right),$$

where $P(\bar{B}_t)$ is the probability that no success arrives before time t.[16] Like in the exogenous-timing setting, the linearity in the latter expression implies that the best response investment level at time t is as follows:

$$k \begin{cases} = 1 & \text{if } \nu_t(\mathbf{K}) > \underline{p}, \\ \in [0,1] & \text{if } \nu_t(\mathbf{K}) = \underline{p}, \\ = 0 & \text{if } \nu_t(\mathbf{K}) < \underline{p}, \end{cases}$$

where the attractiveness of investment is here defined by

$$\nu_t(\mathbf{K}) := F(K_t) p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t K_s ds}.$$

[15]If $n \geq 0$ other players obtain a success at the same time, the player receives a payoff of $1/(n+1)$. Her expected reward is thus $F(K_t) = \sum_{n=0}^\infty \frac{1}{n+1} P(n \text{ other players obtain a success})$. Neglecting the probability that the same player obtains more than one success, the number of players obtaining a success at time t follows the same law as the number of successes occurring at time t, which is a Poisson law of intensity λK_t. Therefore, $F(K_t) = \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{(\lambda K_t)^n}{n!} e^{-\lambda K_t}$, which simplifies to $\frac{1-e^{-\lambda K_t}}{\lambda K_t}$.

[16]Precisely, $P(\bar{B}_t) = 1 - p_0 + p_0 e^{-\lambda \sum_{s=0}^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} K_s ds}$.

As the identity of players is irrelevant, any given investment distribution can be induced by many strategy profiles. We first establish that all atomic equilibria induce the same investment distribution and yield a payoff of 0 to all players.

Lemma 1 (Rent equalization)**.** *If* $\mathbf{K}$ *and* $\mathbf{K}'$ *are atomic-equilibrium investment distributions, then* $\mathbf{K} = \mathbf{K}'$ *and*

$$K_t(\nu_t(\mathbf{K}) - \underline{p}) = 0 \text{ for all } t. \tag{14}$$

Proof. See Section E in the Appendix. □

This is the first major difference from the exogenous timing setting: The positive rent earned by players $t < \tau^*(p_0)$ in the baseline model vanishes when timing is endogenous: All players receive a payoff of 0 in any atomic equilibrium.

The next proposition characterizes the set of atomic equilibria.

Proposition 10 (Equilibrium characterization)**.** *Let* $\mathbf{K}^*$ *be defined as follows:*

- *If* $p_0 \leq \underline{p}$*, then* $K_t^* = 0$ *for all* t*.*
- *If* $p_0 > \underline{p}$*, then* $K_t^* = \begin{cases} I_n & \text{if } t = n\Delta \text{ and } n \leq \bar{n}_e(p_0), \\ 0 & \text{otherwise,} \end{cases}$ *where* $(I_n)_n$ *is the decreasing sequence defined by* $F(I_n) = \underline{p}\left(1 + \frac{1-p_0}{p_0} e^{\lambda(\sum_{m=0}^{n-1} I_k)\mathbb{1}_{n\geq 1}}\right)$ *for each* $n \in \mathbb{N}$ *and* $\bar{n}_e(p_0) := \sup\{n \in \mathbb{N},\ I_n > 0\}$*.*

$\mathbf{K}$ *is an atomic-equilibrium investment distribution if and only if* $\mathbf{K} = \mathbf{K}^*$*.*

Proof. See Section E in the Appendix. □

As in the exogenous timing setting, the attractiveness of investment always remains below p_0. Consequently, no investment occurs in equilibrium when $p_0 \leq \underline{p}$. When $p_0 > \underline{p}$, the investment dynamics is *lumpy*, in the sense that players invest at discrete points in time $t \in \{n\Delta,\ n \in \mathbb{N}\}$ until the public belief reaches $\underline{p}$.[17] Since players wait for Δ time units before reinvesting, the risk of preemption is always 0 in any atomic equilibrium. Therefore, the atomic equilibrium aggregate investment at each time t satisfies

$$F(K_t^*)p_t = \underline{p},$$

implying that players adjust their investment levels downward over time to account for their increasing pessimism. Moreover, players have no strictly profitable deviation from a strategy

[17] If $\bar{n}_e(p_0) = 0$, there is investment only at time 0, which gives a threshold aspect to the equilibrium dynamics.

profile inducing $\mathbf{K}^*$ because investing at time t yields 0 if $t = n\Delta$ for any $n \leq \bar{n}_e(p_0)$, and a strictly negative payoff otherwise. This allows us to state that any atomic strategy profile inducing $\mathbf{K}^*$ is an equilibrium.

Finally, we examine the conditions under which investment breaks occur in equilibrium, i.e., conditions that ensure $\bar{n}_e(p_0) \geq 1$.

Proposition 11 (Investment breaks). *If $p_0 > \underline{p}$, there is at least one investment break if and only if $p_0 > 1/2$ and $\Delta > \underline{\Delta}_e(p_0)$, where $\underline{\Delta}_e(p_0)$ is the value of Δ solving $F(I_1) = 1$.*

Proof. See Section E in the Appendix. □

The interpretation of this condition is as follows. If $p_0 > \underline{p}$, there is an initial mass of investment I_0 at time 0. This mass is determined by the equation $F(I_0) = \underline{p}/p_0$, implying that I_0 decreases with the outcome lag. For small values of Δ, the initial investment I_0 is so large that the updated belief at time Δ falls below the single-player cutoff $\underline{p}$. In this case, investment ceases permanently after time 0. However, if Δ is sufficiently large, the updated belief at time Δ remains above $\underline{p}$, leading to at least one additional investment mass at time $t = \Delta$. This occurs if and only if $\Delta > \underline{\Delta}_e(p_0)$. Finally, for this condition to hold while ensuring $p_0 > \underline{p}$, we must have $p_0 > 1/2$.

6 Concluding remarks

This paper studies a sequential R&D race with outcome lags. We show that such lags generate a countervailing competition effect that can dominate the learning effect, leading to non-monotonicities in investment behavior and explaining the often observed lumpiness in R&D investments. Importantly, these investment non-monotonicities persist even when players learn not only about the technology, but also about the duration of the outcome lag. We further show that the phenomenon remains robust with convex costs and when the timing of investment is endogenous. While our model is necessarily stylized, we conjecture that the core dynamics we identify are likely to persist under further extensions, which we leave for future work and briefly outline below.

Alternative reward schemes. The winner-takes-all assumption is well suited for disruptive innovations that replace older technologies (e.g., cell phones supplanting pagers, or the internet replacing fax machines). Yet a breakthrough often paves the way for a cascade of related innovations. The 1995 discovery of an exoplanet launched a new field of research in contemporary astrophysics, stimulating the development of new instruments and observational facilities.

Therefore, although Michel Mayor and Didier Queloz were the sole recipients of the Nobel Prize for this discovery, many other research teams have benefited, albeit to a lesser extent, from this breakthrough. To better capture this type of innovation, we could assume that each success yields a positive payoff that decreases as the number of successes increases. In this case, player t's expected payoff would depend not only on the current belief (which would reach 1 after the first success) but also on the expected reward in case of success, which would be influenced by the number of prior successes and the level of investment from players in the time between $t - \Delta$ and t. While this would introduce additional complexity to the analysis, the possibility of non-monotonic marginal benefits of investment suggests that investment breaks could still emerge in such a richer setting.

Private signals. Innovating firms often possess private information before embarking on an R&D process, and research generates insights that remain private for some time. To extend our model to incorporate private information, we could assume that players receive conclusive, private signals about the technology. In a good-news setting, where only positive signals provide information, players would stop full investment earlier than $\tau^*(p_0)$ if they receive no signals, as the absence of information would make them more pessimistic about the technology. After $\tau^*(p_0)$, in the absence of a positive signal, players would invest only a fraction of their resources, mirroring the behavior in our original model. Since private signals accelerate the decline of the updated belief, we conjecture that investment breaks would occur more frequently than in the model without private signals. The bad-news setting is more complex because the common belief may not necessarily decrease over time. A player who receives a negative private signal would not invest. This means that, as time passes without public success and without investment stopping altogether, players may grow more or less confident about the technology. The equilibrium evolution of beliefs in this setting is therefore ambiguous and would require deeper analysis.

Stochastic outcome lag. In many applications, it is more realistic to assume that the outcome lag is not only uncertain but also idiosyncratic to each experiment (similar experiments may take different amounts of time to produce results). For example, de Jong et al. (2024) documented a large variance in the development time for new drugs, from 8 months to 223 months from the first human trial to marketing authorization. The discovery of an exoplanet also involves a certain degree of chance: The telescope must be directed toward the appropriate star at a time when a large, thus more easily detectable, planet is orbiting it. A way to incorporate this kind of stochasticity in our model would be to assume that an experiment initiated at time t yields an outcome at time $t + \Delta_t$, where $(\Delta_t)_t$ is a random process. The impact of such stochasticity on players' tradeoffs is not immediately clear. On the one hand, a player risks being preempted by a successor who achieves a success first. On the other hand, a player may also preempt a predecessor whose experiment has not yet produced an outcome. This dual source of uncertainty could alter the equilibrium dynamics in complex ways. We conjecture that it may be possible to identify conditions on the random process $(\Delta_t)_t$ that ensure the existence of investment breaks in equilibrium.

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Appendix

We introduce some notation, which we shall use throughout the Appendix. We define:

- $x_{t-\Delta} := e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}$ and $x_t := e^{-\lambda \int_0^t k_s ds}$;
- $\Omega(p) := \dfrac{1-p}{p}$.

We also recall that

- $p_t = \dfrac{p_0 x_{t-\Delta}}{1-p_0+p_0 x_{t-\Delta}}$, which can be rewritten $\Omega(p_t)x_{t-\Delta} = \Omega(p_0)$;
- $\mu_t = p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t k_s ds} = \dfrac{p_0 x_t}{1-p_0+p_0 x_{t-\Delta}}$.

A Proofs for Section 3: The funding agency problem

A.1 The funding agency objective

The problem of the funding agency is to maximize $W(\mathbf{k}) = \mathrm{E}[(e^{-rT} - \int_0^T e^{-rs}\alpha k_s ds)]$, where T denotes the random arrival time of the first success. Let us first express $W(\mathbf{k})$.

Lemma 2. *Let T denote the random time of arrival of the first success, with $T = +\infty$ if a success never arrives, and let $W(\mathbf{k}) = \mathrm{E}[(e^{-rT} - \int_0^T e^{-rs}\alpha k_s ds)]$. It holds that*

$$W(\mathbf{k}) = \int_0^\infty e^{-rt}(1-p_0+p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds})(-\alpha k_t + \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} p_t)dt.$$

Proof. The probability of no success arriving before t is 1 if the technology is bad, and $e^{-\lambda \int_0^{(t-\Delta)\geq} k_s ds}$ if the technology is good. Therefore, T is distributed according to density $f(t) = p_0 \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} x_{t-\Delta}$, which allows us to write:

$$\begin{aligned} W(\mathbf{k}) &= \int_0^\infty (e^{-rt} - \int_0^t e^{-rs}\alpha k_s ds) p_0 \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} x_{t-\Delta} dt - \int_0^\infty e^{-rs}\alpha k_s ds (1-p_0+p_0 x_\infty), \\ &= \int_0^\infty e^{-rt} p_0 \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} x_{t-\Delta} dt - \int_0^\infty (\int_0^t e^{-rs}\alpha k_s ds) p_0 \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta} x_{t-\Delta} dt \\ &\quad - \int_0^\infty e^{-rs}\alpha k_s ds (1-p_0+p_0 x_\infty). \end{aligned}$$

Applying Fubini's theorem and observing that $\int_s^\infty \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta}x_{t-\Delta}dt = -x_\infty + x_{s-\Delta}$, we obtain:

$$\begin{aligned} W(\mathbf{k}) &= \int_0^\infty e^{-rt}p_0\lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta}x_{t-\Delta}dt - \int_0^\infty e^{-rt}\alpha k_t p_0 x_{t-\Delta}dt - (1-p_0)\int_0^\infty e^{-rt}\alpha k_t dt, \\ &= \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})(-\alpha k_t + \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta}\frac{p_0x_{t-\Delta}}{1-p_0+p_0x_{t-\Delta}})dt, \\ &= \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})(-\alpha k_t + \lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta}p_t)dt. \end{aligned}$$

□

A.2 Proof of Proposition 1

Applying Lemma 2, we can rewrite the funding agency's objective as follows:

$$W(\mathbf{k}) = \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})(-\alpha k_t)dt + \int_\Delta^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})\lambda k_{t-\Delta}p_t dt.$$

As $p_t = p_0x_{t-\Delta}/(1-p_0+p_0x_{t-\Delta})$,

$$W(\mathbf{k}) = \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})(-\alpha k_t)dt + \int_\Delta^\infty e^{-rt}\lambda k_{t-\Delta}p_0x_{t-\Delta}dt.$$

After a convenient change of variable, we can write:

$$\begin{aligned} W(\mathbf{k}) &= \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})(-\alpha k_t)dt + \int_0^\infty e^{-r(t+\Delta)}\lambda k_t p_0 x_t dt, \\ &= \int_0^{+\infty} e^{-rt}k_t\left(-\alpha(1-p_0+p_0x_{t-\Delta}) + \lambda e^{-r\Delta}p_0x_t\right)dt. \end{aligned}$$

The problem of the agency is therefore

$$\max_{(k_t)_t}\int_0^{+\infty} e^{-rt}k_t\lambda e^{-r\Delta}p_0(-\underline{p}\Omega(p_0) + x_t - \underline{p}x_{t-\Delta})dt. \tag{15}$$

We start by proving the existence of an optimal policy $\hat{\mathbf{k}}$ and by describing the necessary conditions satisfied by $\hat{\mathbf{k}}$.

Lemma 3 (Existence and necessary conditions)**.** *The agency problem admits at least one solution $\hat{\mathbf{k}}$, which satisfies*

(i)

$$\hat{k}_t \begin{cases} = 1, & \text{if } \nu_t > 0, \\ \in [0,1], & \text{if } \nu_t = 0, \\ = 0, & \text{if } \nu_t < 0, \end{cases}$$

with

$$\nu_t := e^{-r(t+\Delta)}(-\underline{p}(1-p_0+p_0x_{t-\Delta}) + p_0x_t) - x_t\gamma_t, \tag{16}$$

where γ_t is defined by

(*ii*) $\dot{\gamma}_t = e^{-r(t+\Delta)} p_0(\alpha k_{t+\Delta} - \lambda k_t) + \lambda k_t \gamma_t$ *and* $\lim_{t\to+\infty} \gamma_t = 0$.

Proof. By standard results (see, e.g., Theorem 15 (p. 237) in Seierstad and Sydsaeter (1985)) the problem admits at least one solution $\hat{\mathbf{k}}$. Moreover, by Bokov (2011), if $(\hat{\mathbf{k}}, \hat{x})$ is a solution of the agency problem, then there exists a continuous, piecewise continuously differentiable, function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ with an initial condition $\gamma_0 \in \mathbb{R}$ such that:

(*i*) for any admissible control k, $H(t, \hat{k}_t, \hat{x}_t, \hat{x}_{t-\Delta}, \gamma_t) \geq H(t, k_t, \hat{x}_t, \hat{x}_{t-\Delta}, \gamma_t)$ (Pontryagin maximum principle);

(*ii*) at any point[18] at which γ is differentiable, $\dot{\gamma}_t = -H_{x_t}(t, k_t, x_t, x_{t-\Delta}, \gamma_t) - H_{x_t}(t+\Delta, k_{t+\Delta}, x_{t+\Delta}, x_t, \gamma_{t+\Delta})$ (Euler condition); $\lim_{t\to+\infty} \gamma_t = 0$ (Transversality condition);

where the Hamiltonian of the problem is

$$H(t, k_t, x_t, x_{t-\Delta}, \gamma_t) = e^{-rt} k_t(-\alpha(1 - p_0 + p_0 x_{t-\Delta}) + \lambda e^{-r\Delta} p_0 x_t) - \lambda k_t x_t \gamma_t$$

Necessary conditions are rewritten as follows:

(*i*) $\hat{k}_t \begin{cases} = 1, & \text{if } \nu_t > 0, \\ \in [0,1], & \text{if } \nu_t = 0, \\ = 0, & \text{if } \nu_t < 0, \end{cases}$

with $\nu_t := e^{-r(t+\Delta)}(-\underline{p}(1 - p_0 + p_0 x_{t-\Delta}) + p_0 x_t) - x_t \gamma_t$.

(*ii*) $\dot{\gamma}_t = e^{-r(t+\Delta)} p_0(\alpha k_{t+\Delta} - \lambda k_t) + \lambda k_t \gamma_t$;

□

Differentiating (16) with respect to t, using expression (*ii*) and rearranging, we obtain that, at any point where ν_t is differentiable, the derivative of ν is

$$\dot{\nu}_t = e^{-r(t+\Delta)} \left(-r(-\underline{p}(1 - p_0 + p_0 x_{t-\Delta}) + p_0 x_t) + \lambda \underline{p} p_0 (k_{t-\Delta} \mathbb{1}_{t\geq\Delta} x_{t-\Delta} - k_{t+\Delta} x_t e^{-r\Delta})\right). \quad (17)$$

We now establish some properties of $\hat{\mathbf{k}}$.

Lemma 4. *Let* $\hat{\mathbf{k}}$ *satisfy* (*i*) *and* (*ii*) *in Lemma 3.*

(*a*) $\hat{k}_t = 0$ *for all* $t \Leftrightarrow p_0 \leq \underline{p}$.

[18] In the sequel, the fact that $\dot{\gamma}_t$ and $\dot{\nu}_t$ are defined only at points at which γ and ν are differentiable will be implicit.

(b) *There is no* τ *such that* $\hat{k}_t = 1$ *for all* $t \geq \tau$.

Proof.

We start with the proof of (a).

• Let us first prove that if $\hat{k}_t = 0$ for all t, then $p_0 \leq \underline{p}$. If $\hat{k}_t = 0$ for all t, then $\gamma_t = 0$ for all t by (ii) in Lemma 3. Plugged into (16), we obtain $\nu_t = e^{-r(t+\Delta)}(-\underline{p} + p_0)$ for all t. However, we know by Lemma 3 that $\hat{k}_t = 0$ implies $\nu_t \leq 0$, which thus further implies $p_0 \leq \underline{p}$.

• Let us now prove that $p_0 \leq \underline{p} \Rightarrow \hat{k}_t = 0$ for all t. As x_t is decreasing, $(-\underline{p}\Omega(p_0) + x_t - \underline{p}x_{t-\Delta}) \leq -\underline{p}\Omega(p_0) + (1-\underline{p})x_{t-\Delta} = \underline{p}x_{t-\Delta}(-\Omega(p_t) + \Omega(\underline{p}))$. As p_t is non increasing, if $p_0 \leq \underline{p}$, then $\Omega(p_t) \geq \Omega(\underline{p})$, thus (15) is negative for all $\mathbf{k}$, which implies that the optimal policy is $\hat{k}_t = 0$ for all t.

We continue with the proof of (b). Suppose that there is τ such that $\hat{k}_t = 1$ for all $t \geq \tau$. Then, for all $t \geq \tau + \Delta$, $x_t = x_\tau e^{-\lambda(t-\tau)}$ and $x_{t-\Delta} = x_\tau e^{-\lambda(t-\Delta-\tau)}$. Moreover, for all $t \geq \tau$, $\dot{\gamma}_t = \lambda\gamma_t - e^{-r(t+\Delta)}p_0(\lambda - \alpha)$. Integrating this between τ and $+\infty$ and using the transversality condition in (ii), we obtain that $\gamma_t = \frac{\lambda-\alpha}{r+\lambda}p_0 e^{-r(t+\Delta)}$ for all $t \geq \tau$. Plugging this into (16) for any $t \geq \tau + \Delta$, we find

$$\lambda\nu_t = e^{-rt}\left(-\alpha(1-p_0) + p_0 x_\tau e^{-\lambda(t-\tau)}(-\alpha e^{\lambda\Delta} + \frac{r+\alpha}{\lambda+r}\lambda e^{-r\Delta})\right),$$

which is negative for t large enough, and contradicts the proposition $\hat{k}_t = 1$ for all $t \geq \tau$.

□

We now use Lemma 4 to determine $\int_0^{+\infty} \hat{k}_t dt$. Let $\hat{\mathbf{k}}$ satisfy (i) and (ii) in Lemma 3.

• If $p_0 \leq \underline{p}$, then $\int_0^{+\infty} \hat{k}_t dt = 0$ by Lemma 4-(a).

• Consider now the case $p_0 > \underline{p}$. Let us first prove that $x_{t-\Delta} \geq \frac{\Omega(p_0)}{\Omega(\underline{p})}$ for all t. Recall that the objective of the agency (15) is

$$\int_0^{+\infty} e^{-rt}k_t\lambda e^{-r\Delta}p_0\underbrace{(-\underline{p}\Omega(p_0) + x_t - \underline{p}x_{t-\Delta})}_{:=w_t}dt,$$

and let $w_t := -\underline{p}\Omega(p_0) + x_t - \underline{p}x_{t-\Delta}$. Suppose, ad absurdum, that there is T such that $x_{T-\Delta} < \frac{\Omega(p_0)}{\Omega(\underline{p})}$. As $p_0 > \underline{p}$, $x_0 > \frac{\Omega(p_0)}{\Omega(\underline{p})}$, hence there is τ such that $x_\tau = \frac{\Omega(p_0)}{\Omega(\underline{p})}$, with $x_t < \frac{\Omega(p_0)}{\Omega(\underline{p})}$ for all $t > \tau$. By definition of τ, for all $t \in (\tau, T-\Delta)$, $x_{t-\Delta} > \frac{\Omega(p_0)}{\Omega(\underline{p})}$ and $x_t < \frac{\Omega(p_0)}{\Omega(\underline{p})}$, hence $w_t < 0$ for all $t \in (\tau, T-\Delta]$. Moreover, for all $t > T-\Delta$, $x_t < x_{t-\Delta} < \frac{\Omega(p_0)}{\Omega(\underline{p})}$, thus $w_t < -\underline{p}\Omega(p_0) + (1-\underline{p})x_{t-\Delta} < 0$ for all $t > T-\Delta$. As a consequence, $w_t < 0$ for all $t \geq \tau$, which implies $\hat{k}_t = 0$ for all $t \geq \tau$. But then, $x_t = x_\tau$ for all $t \geq \tau$, which contradicts $x_{T-\Delta} < \frac{\Omega(p_0)}{\Omega(\underline{p})}$.

Let us now prove that $\lim_{t\to+\infty} x_t = \frac{\Omega(p_0)}{\Omega(\underline{p})}$. As $x_{t-\Delta}$ is weakly decreasing and bounded below by $\frac{\Omega(p_0)}{\Omega(\underline{p})}$, it converges to some limit $x^* \geq \frac{\Omega(p_0)}{\Omega(\underline{p})}$. Let us prove that $x^* = \frac{\Omega(p_0)}{\Omega(\underline{p})}$. We rewrite w_t as follows:

$$w_t = -\underline{p}\Omega(p_0) + (1-\underline{p})x^* + (x_t - x^*) - \underline{p}(x_{t-\Delta} - x^*).$$

By definition of the limit, for every $\varepsilon > 0$, there exists T_ε such that $|x_t - x^*| < \varepsilon$ and $|x_{t-\Delta} - x^*| < \varepsilon$ for all $t > T_\epsilon$. Therefore, for every $t > T_\varepsilon$,

$$w_t > -\underline{p}\Omega(p_0) + (1-\underline{p})x^* - \varepsilon(1+\underline{p}).$$

If $x^* > \frac{\Omega(p_0)}{\Omega(\underline{p})}$, then for any $\varepsilon < (-\underline{p}\Omega(p_0) + (1-\underline{p})x^*)/(1+\underline{p})$, $w_t > 0$ for every $t \geq T_\varepsilon$. That implies $\hat{k}_t = 1$ for every $t \geq T_\varepsilon$, which is impossible by Lemma 4-(b). This proves that, if $\int_0^{+\infty} \hat{k}_t dt > 0$, then $\int_0^{+\infty} \hat{k}_t dt = \frac{1}{\lambda} \ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$, which concludes the proof of Proposition 1.

A.3 Proof of Proposition 2

We first need to prove that, if $p_0 > \underline{p}$, it is optimal that players fully invest at the outset of the game.

Lemma 5. *If $p_0 > \underline{p}$, there is $\hat{\tau} > 0$ such that $\hat{k}_t = 1$ for all $t < \hat{\tau}$.*

Proof. We start by proving that, if $p_0 > \underline{p}$, then $\nu_0 \geq 0$. By expression (17), for all $t \leq \Delta$,

$$\dot{\nu}_t = e^{-r(t+\Delta)}(-r(p_0 x_t - \underline{p}) - \lambda \underline{p} p_0 k_{t+\Delta} x_t e^{-r\Delta})$$

Fix $p_0 > \underline{p}$ and suppose to the contrary that $\nu_0 \leq 0$. As 1) ν_t is continuous, and 2) $\dot{\nu}_0 < 0$, there is $\varepsilon > 0$ such that $\nu_t < 0$, thus $k_t = 0$, for all $t \in [0, \varepsilon)$. As a consequence, $\dot{\nu}_t < 0$ on $[0, \varepsilon)$ and $\nu_\varepsilon < 0$. Iterating the argument, it follows that $\nu_t < 0$ for all $t \leq \Delta$. As $x_{t-\Delta} = 1$ for all $t \in [\Delta, 2\Delta]$, the argument can be extended to all $t > \Delta$. This implies $\hat{k}_t = 0 \; \forall \, t$, which, by Lemma 4-($a$), implies $p_0 \leq \underline{p}$, which is a contradiction. Therefore, if $p_0 > \underline{p}$, then $\nu_0 > 0$. As ν_t is continuous, there is $\hat{\tau} > 0$ such that $\nu_t > 0$ on $[0, \hat{\tau})$. This proves the result.

□

Now we prove that investment cannot stop in finite time.

Lemma 6. *If $\int_0^{+\infty} \hat{k}_t dt > 0$, there is no τ such that $\hat{k}_t = 0$ for almost all $t \geq \tau$.*

Proof. Suppose that $\int_0^{+\infty} \hat{k}_t dt > 0$ and that there is a cutoff $\tau > 0$ such that $k_t = 0$ for almost all $t \geq \tau$. Let $\tilde{\tau} := \min\{t \text{ s.t } k_{t'} = 0 \text{ for almost all } t' \geq t\}$.

By Lemma 3, $\nu_{\tilde{\tau}} = 0$, and, for all $t \geq \tilde{\tau}$, $\nu_t \leq 0$ and $\gamma_t = 0$. Plugging into (16), we obtain that, for all $t \geq \tilde{\tau}$, $\nu_t = e^{-r(t+\Delta)}(-\underline{p}(1 - p_0 + p_0 x_{t-\Delta}) + p_0 x_t)$. Plugging $k_{t+\Delta} = 0$ into (17), we thus obtain

$$\dot{\nu}_t = -r\nu_t + \lambda e^{-r(t+\Delta)} \underline{p} p_0 k_{t-\Delta} \mathbb{1}_{t \geq \Delta} x_{t-\Delta} \ \forall\, t \geq \tilde{\tau}.$$

By definition of $\tilde{\tau}$, there is $\bar{t} \in (\tilde{\tau} - \Delta, \tilde{\tau})$ such that $k_{\bar{t}} > 0$. Consequently, $\dot{\nu}_{\bar{t}+\Delta}$ is stricly larger than $-r\nu_{\bar{t}+\Delta}$, which is positive as $\bar{t} + \Delta > \tilde{\tau}$. This contradicts $k\bar{t} + \Delta = 0$. □

Lemma 4-(b) and Lemma 5 together prove that $\hat{\mathbf{k}}$ cannot be weakly increasing for all t. Let us now prove it cannot be weakly decreasing for all t. Suppose ad absurdum that $\hat{\mathbf{k}}$ is decreasing for all $t \in \mathbb{R}_+$. Applying Lemma 5, we know that there is a cutoff $\hat{\tau} > 0$ such that $\hat{k}_t = 1$ for all $t \in [0, \hat{\tau})$. As phases where $\hat{k}_t = 1$ cannot be infinite by Lemma 4-(b), this first phase must be followed by a phase where $\hat{k}_t < 1$. If there is $\tilde{t} > \hat{\tau}$ such that $\hat{k}_{\tilde{t}} = 0$, because $\hat{\mathbf{k}}$ is decreasing it must be the case that $\hat{k}_t = 0$ for all $t \geq \tilde{t}$, which is impossible by Lemma 6. Hence, there is no $\tilde{t}$ such that $\hat{k}_{\tilde{t}} = 0$, which implies that $\hat{k}_t \in (0, 1)$ for all $t > \hat{\tau}$, with $\hat{k}_t \geq \hat{k}_{t'}$ for all $t < t'$. Applying the necessary conditions given in Lemma 3, this implies that $\nu_t = 0$ and $\dot{\nu}_t = 0$ for all $t \geq \hat{\tau}$ which, using (17), yields

$$\hat{k}_{t-\Delta} \mathbb{1}_{t \geq \Delta} x_{t-\Delta} - \hat{k}_{t+\Delta} e^{-r\Delta} x_t = \frac{r}{\lambda p_0 \underline{p}} (-\underline{p}(1 - p_0 + p_0 x_{t-\Delta}) + p_0 x_t) \ \forall\, t \geq \hat{\tau} \tag{18}$$

• If $\hat{\tau} < \Delta$, then (18) is rewritten as follows for all $t \in [\hat{\tau}, \Delta)$:

$$\hat{k}_{t+\Delta} = \frac{r}{\alpha p_0} \left(\frac{\underline{p}}{x_t} - p_0 \right).$$

As $\hat{k}_{t+\Delta} \geq 0$, it holds that $\frac{\underline{p}}{x_t} - p_0 \geq 0$. As $1/x_t \leq e^{\lambda\Delta}$, this further implies $p_0 \leq \underline{p} e^{\lambda\Delta}$. Moreover, $x_t = e^{-\lambda \int_0^t k_s ds}$, hence $\partial(1/x_t)/\partial t = \lambda k_t / x_t > 0$. Therefore, $\hat{\mathbf{k}}$ is strictly increasing on $[\hat{\tau} + \Delta, 2\Delta]$, which contradicts the assumption that $\hat{\mathbf{k}}$ is decreasing.

• If $\hat{\tau} \geq \Delta$, then for $t = \hat{\tau}$, (18) is rewritten

$$e^{-\lambda(\hat{\tau}-\Delta)} - \hat{k}_{t+\Delta} e^{-r\Delta - \lambda\hat{\tau}} = \frac{r}{\lambda p_0 \underline{p}} (-\underline{p}(1 - p_0 + p_0 e^{-\lambda(\hat{\tau}-\Delta)}) + p_0 e^{-\lambda\hat{\tau}}),$$

which yields

$$\hat{k}_{\hat{\tau}+\Delta} = e^{(\lambda+r)\Delta} - \frac{r}{\alpha}(1 - \underline{p} e^{\lambda\Delta} - \underline{p}\Omega(p_0) e^{\lambda\hat{\tau}}).$$

Let us establish that $\hat{k}_{\hat{\tau}+\Delta} > 1$. As $\hat{\tau} > \Delta$,

$$\hat{k}_{\hat{\tau}+\Delta} > e^{(\lambda+r)\Delta} \left(1 + \frac{r}{\lambda p_0} \right) - \frac{r}{\alpha}.$$

Let us now prove that the function $r \mapsto h(r) := e^{(\lambda+r)\Delta} \left(1 + \frac{r}{\lambda p_0} \right) - \frac{r}{\alpha} - 1$ is strictly positive for all r. Differentiating h twice with respect to r, we obtain:

$$h'(r) = e^{(\lambda+r)\Delta}\left(\Delta(1+\frac{r}{\lambda p_0}) + \frac{1}{\lambda p_0}\right) - \frac{1}{\alpha}$$
$$h''(r) = \Delta^2 e^{(\lambda+r)\Delta} + \frac{\Delta}{\alpha p_0}\underline{p}e^{\lambda\Delta}(2+\Delta r)$$

As $h''(r) > 0$ for all r, h' is increasing in r, with $h'(0) = e^{\lambda\Delta}(\Delta + \frac{1}{\lambda p_0}) - \frac{1}{\alpha}$ and $\lim_{r\to+\infty} h'(r) = +\infty$. There are two cases.

• If $h'(0) > 0$, then $h'(r) > 0$ for all $r > 0$. As $h(0) = e^{\lambda\Delta} - 1 > 0$, this implies $h(r) > 0$ for all $r > 0$.

• If $h'(0) \leq 0$, then there is $\tilde{r} \in (0, +\infty)$ such that $h'(\tilde{r}) = 0$, such that $h(r)$ decreases on $[0, \tilde{r}]$ and increases on $[\tilde{r}, +\infty)$. Let us prove that $h(\tilde{r}) > 0$. By definition, $h'(\tilde{r}) = 0$, thus

$$e^{(\lambda+\tilde{r})\Delta} = \frac{1/\alpha}{\Delta(1+\frac{\tilde{r}}{\lambda p_0}) + \frac{1}{\lambda p_0}} \tag{19}$$

Therefore,

$$\begin{aligned} h(\tilde{r}) &= \tfrac{1}{\Delta(1+\frac{\tilde{r}}{\lambda p_0})+\frac{1}{\lambda p_0}}\left(\tfrac{1}{\alpha}(1+\tfrac{r}{\lambda p_0}) - (1+\tfrac{\tilde{r}}{\alpha})(\Delta(1+\tfrac{\tilde{r}}{\lambda p_0}) + \tfrac{1}{\lambda p_0})\right) \\ &= \tfrac{1}{\alpha\lambda p_0}\tfrac{1}{\Delta(1+\frac{\tilde{r}}{\lambda p_0})+\frac{1}{\lambda p_0}}\left(\lambda p_0 - \alpha - (\alpha+\tilde{r})\Delta(\lambda p_0 + \tilde{r})\right) \end{aligned}$$

Let $g(\Delta) := \lambda p_0 - \alpha - (\alpha+\tilde{r})\Delta(\lambda p_0 + \tilde{r})$. Differentiating g with respect to Δ, we obtain:

$$g'(\Delta) = -\frac{\partial\tilde{r}}{\partial\Delta}\Delta(\lambda p_0 + \alpha + 2\tilde{r}) - (\alpha+\tilde{r})(\lambda p_0 + \tilde{r})$$

Differentiating (19) with respect to Δ, we obtain:

$$\frac{\partial\tilde{r}}{\partial\Delta}\Delta = \frac{\lambda(1-p_0)}{\lambda p_0\Delta + 2 + \tilde{r}\Delta} - (\lambda + \tilde{r})$$

Plugging the expression of $\frac{\partial\tilde{r}}{\partial\Delta}$ into the latter expression, we obtain that $g'(\Delta) > 0$. As $g(0) > 0$, this implies $g(\Delta) > 0$ for all Δ, thus $h(\tilde{r}) > 0$.

As a conclusion, $\hat{k}_{\hat{\tau}+\Delta} > 1$, which brings the desired contradiction and proves that $\hat{\mathbf{k}}$ cannot be decreasing.

B Proofs for Section 4: The strategic problem

B.1 Proof of Proposition 3

Recall that the best response of player t to k_{-t} is

$$k_t \begin{cases} = 1 & \text{if } \mu_t > \underline{p}, \\ \in [0,1] & \text{if } \mu_t = \underline{p}, \\ = 0 & \text{if } \mu_t < \underline{p}, \end{cases}$$

with $\mu_t = p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t k_s ds}$. Differentiating μ_t with respect to t, we obtain

$$\dot{\mu}_t = e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t k_s ds} \left(\dot{p}_t - p_t\lambda(k_t - k_{t-\Delta}\mathbb{1}_{t\geq\Delta})\right),$$

which reduces to

$$\dot{\mu}_t = -\lambda\mu_t(k_t - p_t k_{t-\Delta}\mathbb{1}_{t\geq\Delta}) \tag{20}$$

after using (2): $\dot{p}_t = -p_t(1-p_t)\lambda k_{t-\Delta}\mathbb{1}_{t\geq\Delta}$.

We start by proving that there is no investment in equilibrium when $p_0 \leq \underline{p}$.

Lemma 7. *If $p_0 \leq \underline{p}$, then, in the unique equilibrium, $k_t^* = 0$ for all $t > 0$.*

Proof. For every t and every k_{-t} it holds that $\mu_t \leq p_t \leq p_0 = \mu_0$. If $p_0 < \underline{p}$, then $\mu_t < \underline{p}$ for all t, which implies that $k_t = 0$ is dominant for all $t \geq 0$ by (8). If $p_0 = \underline{p}$ and $k_0 > 0$, then $\dot{\mu}_0 < 0$, thus $\mu_t < \underline{p}$ for all $t > 0$, and $k_t = 0$ is dominant for all $t > 0$. □

Next, we prove that when $p_0 > \underline{p}$, in equilibrium there is a cutoff $\tau^*(p_0)$ such that $k_t^* = 1$ for all $t < \tau^*(p_0)$.

Lemma 8. *Let $p_0 > \underline{p}$. In any equilibrium $\mathbf{k}^*$, $k_t^* = 1$ for all $t < \tau^*(p_0)$ where $\tau^*(p_0)$ is defined by $\tau^*(p_0) := \inf\{t \mid \mu_t = \underline{p}\}$, i.e.,*

$$\tau^*(p_0) := \begin{cases} -\frac{1}{\lambda}\ln\left(\frac{\underline{p}}{p_0}\right) & \text{if } p_0 \in [\underline{p}, \underline{p}e^{\lambda\Delta}], \\ \Delta + \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p}e^{\lambda\Delta})}{\Omega(p_0)}\right) & \text{if } p_0 \geq \underline{p}e^{\lambda\Delta}. \end{cases}$$

Proof. Suppose that $p_0 > \underline{p}$. Because $\mu_0 = p_0$, it holds that $\mu_0 > \underline{p}$. Therefore, in equilibrium there exists $\tau > 0$ such that $\mu_s > \underline{p}$ for every $s \leq \tau$, hence such that $k_s^* = 1$ for every $s \leq \tau$ by the best-response condition (8). Plugging this into the expression of μ_t and $\dot{p}_t$, we have

$$\mu_\tau = p_\tau e^{-\lambda(\tau-(\tau-\Delta)\mathbb{1}_{\tau\geq\Delta})} \text{ and } \dot{p}_\tau = -\lambda p_\tau(1-p_\tau)\mathbb{1}_{\tau\geq\Delta}.$$

Straightforwardly, the function $h(\tau) := p_\tau e^{-\lambda(\tau-(\tau-\Delta)\mathbb{1}_{\tau\geq\Delta})}$ is continuous and strictly decreasing in τ. Moreover, h takes the value $p_0 > \underline{p}$ in $\tau = 0$ and the value $\underline{p}e^{-\lambda\Delta} < \underline{p}$ in $\tau = \Delta + \frac{1}{\lambda}\ln\left(\frac{\Omega(\underline{p})}{\Omega(p_0)}\right)$. This implies that there exists a unique value of τ, denoted by $\tau^*(p_0)$, such that $h(\tau) = \underline{p}$, whose expression depends on whether it is larger or smaller than Δ.

- If $h(\Delta) > \underline{p}$, then $\tau^*(p_0) > \Delta$, hence

$$\begin{aligned} h(\tau^*(p_0)) = \underline{p} \quad &\Leftrightarrow p_{\tau^*(p_0)}e^{-\lambda\Delta} = \underline{p} \\ &\Leftrightarrow p_{\tau^*(p_0)} = \underline{p}e^{\lambda\Delta} \end{aligned}$$

Integrating the law of motion of p_t between Δ and $\tau^*(p_0) > \Delta$, we obtain

$$\tau^*(p_0) = \Delta + \frac{1}{\lambda} \ln \left(\frac{\Omega(\underline{p}e^{\lambda\Delta})}{\Omega(p_0)} \right).$$

The condition $\tau^*(p_0) > \Delta$ is thus equivalent to $p_0 > \underline{p}e^{\lambda\Delta}$.

- If $h(\Delta) < \underline{p}$, then $\tau^*(p_0) < \Delta$, hence

$$\begin{aligned} h(\tau^*(p_0)) = \underline{p} \quad &\Leftrightarrow p_0 e^{-\lambda\tau^*(p_0)} = \underline{p} \\ &\Leftrightarrow e^{\lambda(\Delta-\tau^*(p_0))} = \frac{\underline{p}e^{\lambda\Delta}}{p_0} \end{aligned}$$

which is rewritten

$$\tau^*(p_0) = -\frac{1}{\lambda} \ln \left(\frac{\underline{p}}{p_0} \right).$$

The condition $\tau^*(p_0) < \Delta$ is thus equivalent to $p_0 < \underline{p}e^{\lambda\Delta}$. □

Next, we characterize k_t^* for all $t \geq \tau^*(p_0)$.

Lemma 9. *In any equilibrium, if $\mu_t = \underline{p}$, then $\mu_s = \underline{p}$ for every $s \geq t$.*

Proof. Fix some player t and suppose that $\mu_t = \underline{p}$. As μ_t is continuous, if there is $t' > t$ such that $\mu_{t'} > \underline{p}$, then there is an open interval $S \subset [t, t']$ such that $\dot{\mu}_s > 0$ and $\mu_s > \underline{p}$ for every $s \in S$. This implies that $k_s = 1$ for every $s \in S$ by (8), which contradicts $\dot{\mu}_s > 0$ by (20). Also, if there is $t' > t$ such that $\mu_{t'} < \underline{p}$, then there is an open interval $S \subset [t, t']$ such that $\dot{\mu}_s < 0$ and $\mu_s < \underline{p}$ for every $s \in S$. This implies that $k_s = 0$ for every $s \in S$ by (8), thus contradicts $\dot{\mu}_s < 0$ by (20).

□

Applying Lemma 8, $k_t^* = 1$ for all $t < \tau^*(p_0)$ and $\mu_{\tau^*(p_0)} = \underline{p}$. Applying Lemma 9, $\mu_t = \underline{p}$ for all $t \geq \tau^*(p_0)$. As μ_t is constant on $[\tau^*(p_0), +\infty)$ in any equilibrium, its derivative is $\dot{\mu}_t = 0$ for every $t \geq \tau(p_0)$. By (20), this implies that

$$k_t^* = p_t k_{t-\Delta}^* \mathbb{1}_{t \geq \Delta} \text{ for almost all } t \geq \tau^*(p_0). \tag{21}$$

We have proved that any equilibrium $\mathbf{k}^*$ satisfies $k_t^* = 1$ if $t < \tau^*(p_0)$ and $k_t^* = p_t k_{t-\Delta}^* \mathbb{1}_{t \geq \Delta}$ for almost all $t \geq \tau^*(p_0)$. As p_t is determined by $(k_s^*)_{s<t}$, the equilibrium action at t depends on all equilibrium actions before t. Moreover, in any equilibrium, $k_0^* = 1$ if $p_0 > \underline{p}$. Therefore, there exists a unique function $\mathbf{k}^*$ satisfying the equilibrium conditions, thus there is an essentially unique equilibrium. This concludes the proof of Proposition 3.

B.2 Proof of Proposition 4

The aim of this (tedious) proof is to determine the analytical expression of k_t^*. For ease of notation, τ will stand for $\tau^*(p_0)$ throughout the proof. We also denote by p_t^n and k_t^n the public belief and player t's action at time $t \in [\tau + (n-1)\Delta, \tau + n\Delta]$, and write $\tilde{p}_n := p^n_{\tau+(n-1)\Delta} = p^{n-1}_{\tau+(n-1)\Delta}$. By definition of τ,

$$\tilde{p}_1 = p_\tau = \begin{cases} p_0, & \text{if } \tau \leq \Delta, \\ \underline{p}e^{\lambda\Delta} & \text{if } \tau \geq \Delta. \end{cases}$$

Note that in the case where $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta})$, $k_t^n = 0$ and $p_t^n = \tilde{p}_n$ for all $t \in [\tau + (n-1)\Delta, n\Delta]$. The differences between the cases $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta})$ and $p_0 \geq p_0 e^{\lambda\Delta}$ are illustrated in Figures 3 and 4.

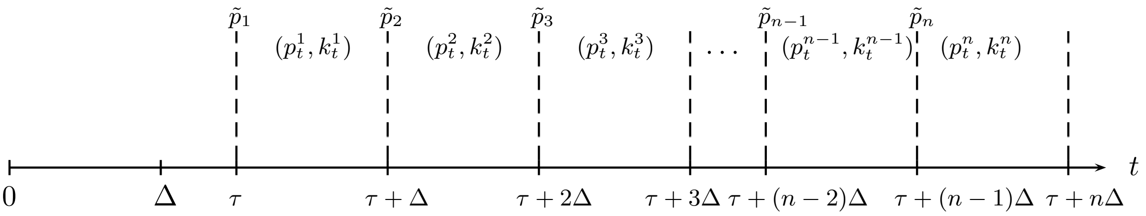

Figure 3: $p_0 \geq \underline{p}e^{\lambda\Delta}$.

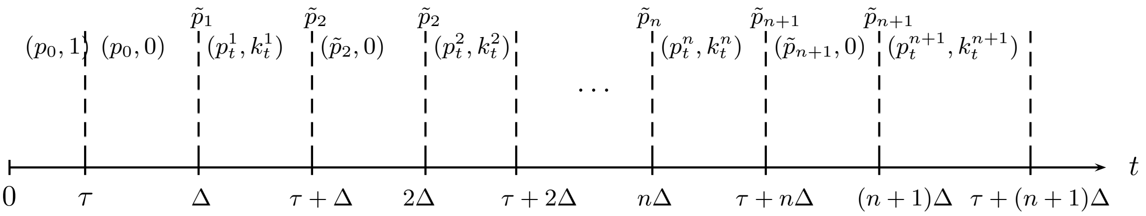

Figure 4: $p_0 \in (\underline{p}, \underline{p}e^{\lambda\Delta})$.

Step 1 We shall first establish by induction that, for every $n \geq 2$, and for all $t \geq \Delta$,

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}_n)} = \frac{1}{1 - (1 - e^{-\lambda(t-(n-1)\Delta-\max\{\tau,\Delta\})}) \prod_{k=1}^{n-1} \tilde{p}^k} \tag{22}$$

Let us first express p_t^n as a function of $p_{t-\Delta}^{n-1}$. The law of motion of the common belief is $\dot{p}_t = -\lambda p_t(1-p_t)k_{t-\Delta}\mathbb{1}_{t\geq\Delta}$ for all t. In particular, $\dot{p}_{t-\Delta} = -\lambda p_{t-\Delta}(1-p_{t-\Delta})k_{t-2\Delta}\mathbb{1}_{t\geq 2\Delta}$ (a)

holds for every t. Moreover, applying the indifference condition (21) to $t' = t-\Delta$, in equilibrium, it holds that $k_{t-\Delta}\mathbb{1}_{t\geq\Delta} = p_{t-\Delta}k_{t-2\Delta}\mathbb{1}_{t\geq 2\Delta}$ (b). Plugging (b) into (a), we obtain:

$$k_{t-\Delta}\mathbb{1}_{t\geq\Delta} = -\frac{\dot{p}_{t-\Delta}}{\lambda(1-p_{t-\Delta})}\mathbb{1}_{t\geq\Delta}.$$

Now, fix some $n \geq 2$. For every $t \in [\tau+(n-1)\Delta, \tau+n\Delta)$, $\dot{p}_t^n = -\lambda p_t^n(1-p_t^n)k_{t-\Delta}^{n-1}\mathbb{1}_{t\geq\Delta}$ by definition and we have just shown that $k_{t-\Delta}^{n-1}\mathbb{1}_{t\geq\Delta} = -\frac{\dot{p}_{t-\Delta}^{n-1}}{\lambda(1-p_{t-\Delta}^{n-1})}\mathbb{1}_{t\geq\Delta}$. Combining these two expressions and rearranging, we obtain:

$$\frac{\dot{p}_t^n}{p_t^n(1-p_t^n)} = \frac{\dot{p}_{t-\Delta}^{n-1}}{1-p_{t-\Delta}^{n-1}}. \tag{23}$$

Integrating (23) between $\tau+(n-1)\Delta$ and some $t \in [\tau+(n-1)\Delta, \tau+n\Delta)$, we obtain:

$$\ln\left(\frac{\Omega(p_t^n)}{\Omega(p_{\tau+(n-1)\Delta}^n)}\right) = \ln\left(\frac{1-p_{t-\Delta}^{n-1}}{1-p_{\tau+(n-2)\Delta}^{n-1}}\right),$$

which, by definition of $\tilde{p}_n$, can be rewritten:

$$\frac{\Omega(p_t^n)}{\Omega(\tilde{p}_n)} = \frac{1-p_{t-\Delta}^{n-1}}{1-\tilde{p}_{n-1}}. \tag{24}$$

Let us now prove that (22) is true for $n=2$. As $k_t = 1$ for every $t \leq \tau$, $\Omega(p_t^1) = \Omega(\tilde{p}_1)e^{\lambda(t-\max\{\tau,\Delta\})}$ for all $t \geq \Delta$, i.e.,

$$1-p_t^1 = \frac{\Omega(\tilde{p}_1)}{\Omega(\tilde{p}_1)+e^{-\lambda(t-\max\{\tau,\Delta\})}} = \frac{1-\tilde{p}_1}{1-\tilde{p}_1+\tilde{p}_1 e^{-\lambda(t-\max\{\tau,\Delta\})}}.$$

Plugging $n=2$ into (24) yields $\frac{\Omega(p_t^2)}{\Omega(\tilde{p}_2)} = \frac{1-p_{t-\Delta}^1}{1-\tilde{p}_1}$. Therefore,

$$\frac{\Omega(p_t^2)}{\Omega(\tilde{p}_2)} = \frac{1}{1-(1-e^{-\lambda(t-\Delta-\max\{\tau,\Delta\})})\tilde{p}_1},$$

which implies that (22) is true for $n=2$.

Let us now suppose that (22) is true for some given n, and let us prove that it is true for $n+1$. Again, by (24) it holds that $\dfrac{\Omega(p_t^{n+1})}{\Omega(\tilde{p}_{n+1})} = \dfrac{1-p_{t-\Delta}^n}{1-\tilde{p}_n}$. Let us use (22) to derive $1-p_{t-\Delta}^n$. If (22) holds for n, then

$$1-p_t^n = \frac{\Omega(\tilde{p}_n)}{\Omega(\tilde{p}_n)+1-(1-e^{-\lambda(t-(n-1)\Delta-\max\{\tau,\Delta\})})\prod_{k=1}^{n-1}\tilde{p}_k},$$

which implies that

$$1-p_{t-\Delta}^n = \frac{\Omega(\tilde{p}_n)}{\Omega(\tilde{p}_n)+1-(1-e^{-\lambda(t-n\Delta-\max\{\tau,\Delta\})})\prod_{k=1}^{n-1}\tilde{p}_k}.$$

Using (24) and the latter expression, we can write that

$$\begin{aligned}\frac{\Omega(p_t^{n+1})}{\Omega(\tilde{p}_{n+1})} &= \frac{\Omega(\tilde{p}_n)/(1-\tilde{p}_n)}{\Omega(\tilde{p}_n)+1-(1-e^{-\lambda(t-n\Delta-\max\{\tau,\Delta\})})\prod_{k=1}^{n-1}\tilde{p}_k}, \\ &= \frac{1}{1-(1-e^{-\lambda(t-n\Delta-\max\{\tau,\Delta\})})\prod_{k=1}^{n}\tilde{p}_k},\end{aligned}$$

hence (22) is true for $n+1$.

Step 2 Next we shall establish by induction that, for every $n \geq 1$,

$$\Omega(\tilde{p}_n) = \frac{(1-\tilde{p}_1)(1-\underline{p})}{\underline{p}^{n-1}(\tilde{p}_1-\underline{p})+\underline{p}(1-\tilde{p}_1)}. \tag{25}$$

This is straightforward for $n = 1$. Let us prove that (25) holds for $n = 2$. By definition of $\tilde{p}_2$, $\Omega(\tilde{p}_2) = \Omega(p^2_{\tau+\Delta}) = \Omega(p^1_{\tau+\Delta})$. Moreover, since $k_t = 1$ for every $t \leq \tau$, the value of $p^1_{\tau+\Delta}$ depends on whether τ is larger or smaller than Δ, and is given by

$$\Omega(p^1_{\tau+\Delta}) = \begin{cases} \Omega(p_0)e^{\lambda\tau} & \text{if } \tau < \Delta, \\ \Omega(\underline{p}e^{\lambda\Delta})e^{\lambda\Delta} & \text{if } \tau > \Delta. \end{cases}$$

Recall that $\tau^*(p_0) = \frac{1}{\lambda}\ln\left(\frac{p_0}{\underline{p}}\right)$ when $\tau^*(p_0) < \Delta$. As a consequence, the two latter expressions reduce to

$$\Omega(p^1_{\tau+\Delta}) = \begin{cases} \frac{1-p_0}{\underline{p}} & \text{if } \tau < \Delta, \\ \frac{1-\underline{p}e^{\lambda\Delta}}{\underline{p}} & \text{if } \tau > \Delta, \end{cases}$$

which, by definition of $\tilde{p}_1$, can be rewritten as follows:

$$\Omega(p^1_{\tau+\Delta}) = \frac{1-\tilde{p}_1}{\underline{p}}.$$

It is straightforward to verify that plugging $n = 2$ into (25) also yields $\Omega(p^1_{\tau+\Delta}) = \frac{1-\tilde{p}_1}{\underline{p}}$, which proves that (25) is true for $n = 2$.

Let us now fix some $n \geq 3$ and suppose that (25) is true for every $k \leq n$. Our aim is to establish that (25) is true for $n+1$. As $\tilde{p}_{n+1} = p^{n+1}_{\tau+n\Delta} = p^n_{\tau+n\Delta}$ by definition, the value of $\Omega(\tilde{p}_{n+1})$ is obtained by plugging $t = \tau + n\Delta$ into (22), which yields

$$\frac{\Omega(\tilde{p}_{n+1})}{\Omega(\tilde{p}_n)} = \frac{1}{1-(1-e^{-\lambda(\tau+\Delta-\max\{\tau,\Delta\})})\prod_{k=1}^{n-1}\tilde{p}_k}$$

Now, observe that

$$1-e^{-\lambda(\tau+\Delta-\max\{\tau,\Delta\})} = \begin{cases} 1-e^{-\lambda\tau} & \text{if } \tau < \Delta, \\ 1-e^{-\lambda\Delta} & \text{if } \tau \geq \Delta. \end{cases}$$

If $\tau < \Delta$, $1-e^{-\lambda\tau} = 1-\frac{\underline{p}}{p_0} = \frac{\tilde{p}_1-\underline{p}}{\tilde{p}_1}$ since $\tilde{p}_1 = p_0$ when $\tau < \Delta$. Furthermore, if $\tau > \Delta$, $1-e^{-\lambda\Delta} = 1-\frac{\underline{p}}{\underline{p}e^{\lambda\Delta}} = \frac{\tilde{p}_1-\underline{p}}{\tilde{p}_1}$ since $\tilde{p}_1 = \underline{p}e^{\lambda\Delta}$ when $\tau > \Delta$. Therefore,

$$1-e^{-\lambda(\tau+\Delta-\max\{\tau,\Delta\})} = \frac{\tilde{p}_1-\underline{p}}{\tilde{p}_1},$$

which implies that

$$\Omega(\tilde{p}_{n+1}) = \frac{\tilde{p}_1 \Omega(\tilde{p}_n)}{\tilde{p}_1 - (\tilde{p}_1 - \underline{p}) \prod_{k=1}^{n-1} \tilde{p}_k}. \tag{26}$$

Let us now compute $\prod_{k=1}^{n-1} \tilde{p}_k$ under the induction hypothesis. If (25) is true for every $k \leq n$, then for any $k \leq n$,

$$\tilde{p}_k = \frac{\underline{p}^{k-1}(\tilde{p}_1 - \underline{p}) + \underline{p}(1 - \tilde{p}_1)}{\underline{p}^{k-1}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1}.$$

We thus observe that $\tilde{p}_k = \underline{p} \times \dfrac{A(k-1)}{A(k)}$, with $A(k) := \underline{p}^{k-1}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1$. Therefore, the product $\prod_{k=1}^{n-1} \tilde{p}_k$ can be simplified as follows:

$$\begin{aligned} \prod_{k=1}^{n-1} \tilde{p}_k &= \prod_{k=1}^{n-1} \underline{p} \times \frac{A(k-1)}{A(k)} = \underline{p}^{n-1} \frac{A(0)}{A(n-1)} \\ &= \underline{p}^{n-1} \frac{\underline{p}^{-1}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1}{\underline{p}^{n-2}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1} \end{aligned}$$

After a last simplification, we obtain

$$\prod_{k=1}^{n-1} \tilde{p}_k = \underline{p}^{n-2} \frac{\tilde{p}_1(1 - \underline{p})}{\underline{p}^{n-2}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1}. \tag{27}$$

Plugging this into (26) and simplifying by $\tilde{p}_1$, we obtain:

$$\Omega(\tilde{p}^{n+1}) = \Omega(\tilde{p}^n) \frac{\underline{p}^{n-2}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1}{\underline{p}^{n-2}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1 - (\tilde{p}_1 - \underline{p})\underline{p}^{n-2}(1 - \underline{p})}$$

As $\Omega(\tilde{p}^n) = \dfrac{(1 - \tilde{p}_1)(1 - \underline{p})}{\underline{p}} \dfrac{1}{\underline{p}^{n-2}(\tilde{p}_1 - \underline{p}) + 1 - \tilde{p}_1}$ under the induction hypothesis, we further obtain:

$$\Omega(\tilde{p}^{n+1}) = \frac{(1 - \tilde{p}_1)(1 - \underline{p})}{\underline{p}^n(\tilde{p}_1 - \underline{p}) + \underline{p}(1 - \tilde{p}_1)},$$

which proves that (25) is true for $n + 1$.

Step 3 We can now derive the expression of k_t^n. As $k_t = 1$ if $t \leq \tau$ and $k_t = p_t k_{t-\Delta} \mathbb{1}_{t \geq \Delta}$ if $t \geq \tau$, it is straightforward to prove by induction that, for all $t \geq \Delta$,

$$k_t^n = \prod_{i=0}^{n-1} p_{t-i\Delta}^{n-i}.$$

Plugging (27) and (25) into (22), we obtain:

$$\Omega(p_t^n) = \frac{\Omega(\underline{p})\Omega(\tilde{p}_1)}{\Omega(\tilde{p}_1)(1 - \underline{p}^{n-1}) + \underline{p}^{n-2}(1 - \underline{p})e^{-\lambda(t-(n-1)\Delta - \max\{\tau,\Delta\})}},$$

which can be rearranged as follows:

$$p_t^n = \underline{p} \frac{\Omega(\tilde{p}_1)(1 - \underline{p}^{n-1}) + \underline{p}^{n-2}(1 - \underline{p})e^{-\lambda(t-(n-1)\Delta - \max\{\tau,\Delta\})}}{\Omega(\tilde{p}_1)(1 - \underline{p}^n) + \underline{p}^{n-1}(1 - \underline{p})e^{-\lambda(t-(n-1)\Delta - \max\{\tau,\Delta\})}}.$$

Because $t - i\Delta - (n-i-1)\Delta - \max\{\tau, \Delta\} = t - (n-1)\Delta - \max\{\tau, \Delta\}$, for every $i \leq n-2$,

$$p^{n-i}_{t-i\Delta} = \underline{p}\frac{B(i+1;t)}{B(i;t)},$$

with $B(i;t) := \Omega(\tilde{p}_1)(1-\underline{p}^{n-i}) + \underline{p}^{n-1-i}(1-\underline{p})e^{-\lambda(t-(n-1)\Delta-\max\{\tau,\Delta\})}$. Therefore,

$$k^n_t = \prod_{i=0}^{n-1}\left(\underline{p}\frac{B(i+1;t)}{B(i;t)}\right) = \underline{p}^n\frac{B(n;t)}{B(0;t)},$$

which simplifies to:

$$k^n_t = \frac{\Omega(\underline{p})}{\Omega(\underline{p}) + \Omega(\tilde{p}_1)\Omega(\underline{p}^n)e^{\lambda(t-(n-1)\Delta-\max\{\tau,\Delta\})}}$$

Finally, one observes that

$$\Omega(\tilde{p}_1)e^{\lambda(\Delta-\max\{\tau,\Delta\})} = \begin{cases} \Omega(p_0), & \text{if } \tau < \Delta, \\ \Omega(\underline{p}e^{\lambda\Delta})e^{\lambda(\Delta-\tau)}, & \text{if } \tau \geq \Delta. \end{cases}$$

As, by definition of $\tau^*(p_0)$, $\Omega(\underline{p}e^{\lambda\Delta}) = \Omega(p_0)e^{\lambda(\tau-\Delta)}$ when $\tau > \Delta$, it turns out that $\Omega(\tilde{p}_1)e^{\lambda(\Delta-\max\{\tau,\Delta\})} = \Omega(p_0)$, hence that

$$k^n_t = \frac{\Omega(\underline{p})}{\Omega(\underline{p}) + \Omega(p_0)\Omega(\underline{p}^n)e^{\lambda(t-n\Delta)}}.$$

The observation that $\Omega(k^n_t) = \frac{1-k^n_t}{k^n_t} = \frac{\Omega(p_0)\Omega(\underline{p}^n)e^{\lambda(t-n\Delta)}}{\Omega(\underline{p})}$ concludes the proof of Proposition 4.

B.3 Proof of Proposition 5

Using expression (15), the funding agency's payoff can be written as follows:

$$W(\mathbf{k}) = \int_0^\infty e^{-rt}k_t(1-p_0+p_0x_{t-\Delta})(\frac{p_0x_t}{1-p_0+p_0x_{t-\Delta}}\lambda e^{-r\Delta} - \alpha)dt,$$

which, using the player's payoff expression (7), gives

$$W(\mathbf{k}) = \int_0^\infty e^{-rt}(1-p_0+p_0x_{t-\Delta})u(k_t;k_{-t})dt.$$

We know that, in equilibrium, $k^*_t = 1$ if $t < \tau^*(p_0)$ and $u(k^*_t, k^*_{-t}) = 0$ for all $t \geq \tau^*(p_0)$. Therefore,

$$W(\mathbf{k}^*) = \int_0^{\tau^*(p_0)} e^{-rt}(1-p_0+p_0e^{-\lambda(t-\Delta)\mathbb{1}_{t\geq\Delta}})\lambda e^{-r\Delta}(\mu_t - \underline{p})dt = W(\tilde{\mathbf{k}}),$$

where $\tilde{\mathbf{k}}$ is the cutoff strategy defined by $\tilde{k}_t = \mathbb{1}_{t<\tau^*(p_0)}$. By Lemma 6, $W(\tilde{\mathbf{k}}) < W(\hat{\mathbf{k}})$. Therefore, $W(\mathbf{k}^*) < W(\hat{\mathbf{k}})$, which proves the desired result.

C Proofs for Section 5.1: Uncertain outcome lag

C.1 Payoff expression

Recall that $\bar{\mathcal{B}}_t$ stands for the event "no breakthrough has occurred by time t" and let Y_t denote the event "player t obtains payoff 1". As player t does not know the outcome lag, her expected payoff at time t is:

$$u(k_t, k_{-t}) = -\alpha k_t + \mathrm{E}[e^{-r\Delta}\mathbb{1}_{Y_t} \mid \bar{\mathcal{B}}_t].$$

We can write

$$\mathrm{E}[e^{-r\Delta}\mathbb{1}_{Y_t} \mid \bar{\mathcal{B}}_t] = \int e^{-r\delta} P(Y_t \mid \Delta = \delta, \bar{\mathcal{B}}_t) f(\delta \mid \bar{\mathcal{B}}_t) d\delta,$$

where

$$P(Y_t \mid \Delta = \delta, \bar{\mathcal{B}}_t) = \lambda k_t \frac{p_0 e^{-\lambda \int_0^{(t-\delta)\mathbb{1}_{t\geq\delta}} k_s ds}}{1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\delta)\mathbb{1}_{t\geq\delta}} k_s ds}} e^{-\lambda \int_{(t-\delta)\mathbb{1}_{t\geq\delta}}^t k_s ds},$$

and

$$f(\delta \mid \bar{\mathcal{B}}_t) = (1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\delta)\mathbb{1}_{t\geq\delta}} k_s ds}) \frac{f(\delta)}{P(\bar{\mathcal{B}}_t)}.$$

Therefore,

$$\mathrm{E}[e^{-r\Delta}\mathbb{1}_{Y_t} \mid \bar{\mathcal{B}}_t] = \frac{\lambda k_t e^{-\lambda \int_0^t k_s ds}}{P(\bar{\mathcal{B}}_t)} \mathrm{E}[e^{-r\Delta}].$$

As the probability that no breakthrough occurs before t is:

$$P(\bar{\mathcal{B}}_t) = 1 - p_0 + p_0 E[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}],$$

we obtain:

$$u(k_t; k_{-t}) = -\alpha k_t + \lambda k_t \frac{p_0 e^{-\lambda \int_0^t k_s ds}}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]} \mathrm{E}[e^{-r\Delta}].$$

C.2 Proof of Proposition 6

Recall that the attractiveness of investment is

$$\xi_t = \frac{p_0 e^{-\lambda \int_0^t k_s ds}}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]}.$$

Differentiating ξ_t with respect to t and simplifying, we obtain:

$$\dot{\xi}_t = -\lambda \xi_t \left(k_t - \frac{p_0 \mathrm{E}[k_{t-\Delta} \mathbb{1}_{t\geq\Delta} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]} \right).$$

We start by proving the following Lemma.

Lemma 10. *For every t,*

(*i*) $\xi_t \leq p_0$ *for all* t.

(*ii*) *If* $k_t = 1$ *and* $p_0 < 1$, *then* $\dot{\xi}_t < 0$.

(*iii*) *If* $k_t = 0$, *then* $\dot{\xi}_t \geq 0$.

Proof. $\forall\ \delta \in [\underline{\Delta}, \overline{\Delta}]$, $e^{-\lambda \int_0^{(t-\delta)\mathbb{1}_{t\geq\delta}} k_s ds} \geq e^{-\lambda \int_0^t k_s ds}$, thus $\mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}] \geq e^{-\lambda \int_0^t k_s ds}$. As, in addition, $1 \geq e^{-\lambda \int_0^t k_s ds}$, it holds that

$$1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}] \geq e^{-\lambda \int_0^t k_s ds},$$

which proves (*i*).

Let $z_t := \frac{p_0 \mathrm{E}[k_{t-\Delta}\mathbb{1}_{t\geq\Delta} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]}{1-p_0+p_0\mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds}]}$. As $k_{t-\delta}\mathbb{1}_{t\geq\delta} \leq 1$ for every $\delta \in [\underline{\Delta}, \overline{\Delta}]$ and $p_0 < 1$, it holds that $0 \leq z_t < 1$. If $k_t = 1$, then $\dot{\xi}_t = -\lambda \xi_t (1 - z_t)$, which is thus strictly negative. This proves (*ii*). If $k_t = 0$, then $\dot{\xi}_t = \lambda \xi_t z_t$, which is non negative. This proves (*iii*). □

We now turn to the proof of Proposition 6. If $p_0 \leq \underline{p}$, $\xi_t \leq \underline{p}$ for every t by Lemma 10. As a consequence, $k_t^* = 0$ for every t. Suppose now $p_0 > \underline{p}$. As $\xi_0 = p_0$, in that case $\xi_0 > \underline{p}$, hence $k_t^* = 1$ in a right neighborhood of 0. By Lemma 10, ξ_t strictly decreases. If $k_t^* = 1$ for all t, then $\lim_{t\to+\infty} \xi_t = 0 < \underline{p}$, thus there is a cutoff $\tau_u^*(p_0) > 0$ such that $\xi_t > \underline{p}$ for all $t \in [0, \tau_u^*(p_0)]$, and $\xi_{\tau_u^*(p_0)} = \underline{p}$, i.e.,

$$\frac{p_0 e^{\lambda \tau_u^*(p_0)}}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(\tau_u^*(p_0)-\Delta)\mathbb{1}_{\tau^*\geq\Delta}} k_s ds}]} = \underline{p}. \tag{28}$$

Let us now prove that, in equilibrium, $\xi_t = \underline{p}$ for every $t \geq \tau_u^*(p_0)$. If $\xi_{\tau_u^*(p_0)+dt} > \underline{p}$, then $k_{\tau_u^*(p_0)+dt} = 1$. By Lemma 10, this implies that $\dot{\xi}_t$ strictly decreases in $\tau_u^*(p_0) + dt$, which contradicts $\xi_{\tau_u^*(p_0)+dt} > \underline{p}$. Therefore, $\xi_{\tau_u^*(p_0)+dt} \leq \underline{p}$. If $\xi_{\tau_u^*(p_0)+dt} < \underline{p}$, then $k_{\tau_u^*(p_0)+dt} = 0$. By Lemma 10, this implies that $\dot{\xi}_t$ increases in $\tau_u^*(p_0)$, which contradicts $\xi_{\tau_u^*(p_0)+dt} < \underline{p}$. This proves that $\xi_{\tau_u^*(p_0)+dt} = \underline{p}$, and that $\xi_t = \underline{p}$ for every $t \geq \tau_u^*(p_0)$. As a consequence, the best response of each player after $\tau_u^*(p_0)$ is given by the indifference condition $\dot{\xi}_t = 0$ for all $t \geq \tau_u^*(p_0)$, which yields,

$$k_t^* = \frac{p_0 \mathrm{E}[k_{t-\Delta}^* \mathbb{1}_{t\geq\Delta} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta} k_s^* ds}}]}{1 - p_0 + p_0 \mathrm{E}[e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s^* ds}]}.$$

C.3 Proof of Proposition 7

Applying Proposition 6, we can state that

$$k_t^* = 0 \Leftrightarrow t \geq \tau_u^*(p_0) \text{ and } k_{t-\delta}\mathbb{1}_{t\geq\delta} = 0\ \forall\ \delta \in [\underline{\Delta}, \overline{\Delta}). \tag{29}$$

A direct consequence of (29) is that if $k_t^* = 0$ for all $t \in [x, y)$, with $k_y^* > 0$ and $\overline{\Delta} - \underline{\Delta} < y - x$, then $k_t^* = 0$ for all $t \in [x + \overline{\Delta}, y + \underline{\Delta})$ and $k_{y+\underline{\Delta}}^* > 0$.

Suppose that $\tau_u^*(p_0) \leq \underline{\Delta}$, and fix some $t \in [\tau_u^*(p_0), \underline{\Delta})$. As $t < \underline{\Delta}$, $\mathbb{1}_{t \geq \delta} = 0$ for all $\delta \in [\underline{\Delta}, \overline{\Delta}]$, thus $\mathrm{E}[k_{t-\Delta}^* \mathbb{1}_{t\geq\Delta} e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s^* ds}] = 0$. Therefore,

$$k_t^* = 0 \text{ for all } t \in [\tau_u^*(p_0), \underline{\Delta}) \text{ and } k_{\underline{\Delta}}^* > 0.$$

Now, for each $n \in \mathbb{N}$, we let $I_n := [\tau_u^*(p_0) + (n-1)\overline{\Delta}, n\underline{\Delta})$. We have proved that, if $\tau_u^*(p_0) < \underline{\Delta}$, then $k_t^* = 0$ everywhere on I_1 and $k_{\underline{\Delta}}^* > 0$. Applying (29), if $2\underline{\Delta} > \tau_u^*(p_0) + \overline{\Delta}$, then $k_t^* = 0$ everywhere on I_2. Going on iteratively, for each $n \in \mathbb{N}$, if $k_t^* = 0$ everywhere on I_n, with $k_{n\underline{\Delta}}^* > 0$ and $(n+1)\underline{\Delta} > \tau_u^*(p_0) + n\overline{\Delta}$, then $k_t^* = 0$ everywhere on I_{n+1} and $k_{(n+1)\underline{\Delta}}^* > 0$.

Let $\bar{n}_u(p_0) \in \mathbb{N}$ such that $\bar{n}_u(p_0) < \frac{\overline{\Delta} - \tau_u^*(p_0)}{\overline{\Delta} - \underline{\Delta}}$ and $\bar{n}_u(p_0) + 1 \geq \frac{\overline{\Delta} - \tau_u^*(p_0)}{\overline{\Delta} - \underline{\Delta}}$. Note that, because $\tau_u^*(p_0) < \underline{\Delta}$, $\bar{n}_u(p_0)$ exists and $\bar{n}_u(p_0) \geq 1$. By definition of $\bar{n}_u(p_0)$, $k_t^* = 0$ for all $t \in I_{\bar{n}_u(p_0)}$, and $k_t^* > 0$ for all $t \geq \bar{n}_u(p_0)\underline{\Delta}$.

D Proofs for Section 5.2: Convex investment costs

D.1 Proof of Proposition 9

We start by proving that, in equilibrium, both the investment level and the common belief converge to 0.

Lemma 11. *Let* $\mathbf{k}$ *be an equilibrium. If* $p_0 < 1$, $\lim_{t\to\infty} p_t = \lim_{t\to\infty} k_t^* = 0$.

Proof. If $p_0 < 1$, the function $t \mapsto p_t$ is weakly decreasing and bounded below by 0, thus it converges towards some limit $p^l \geq 0$, with $p_t \geq p^l$ for all t. As $k_t^* = \mu_t/\underline{p}$ and μ_t is bounded above by p_0, $k_t^* \leq p_0/\underline{p}$ for all t. Moreover, $\mu_t = p_t e^{-\lambda \int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^t k_s ds}$. Therefore, μ_t is bounded below by $p^l e^{-\lambda\Delta \frac{p_0}{\underline{p}}}$, which further implies that $k_t^* \geq \frac{p^l}{\underline{p}} e^{-\lambda\Delta\frac{p_0}{\underline{p}}}$ for all t. Since $p_t = (p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds})/(1 - p_0 + p_0 e^{-\lambda \int_0^{(t-\Delta)\mathbb{1}_{t\geq\Delta}} k_s ds})$, the latter inequality implies that

$$p_t \leq \frac{p_0 e^{-\lambda(t-\Delta)\mathbb{1}_{t\geq\Delta}\frac{p^l}{\underline{p}} e^{-\lambda\Delta\frac{p_0}{\underline{p}}}}}{1 - p_0 + p_0 e^{-\lambda(t-\Delta)\mathbb{1}_{t\geq\Delta}\frac{p^l}{\underline{p}} e^{-\lambda\Delta\frac{p_0}{\underline{p}}}}}.$$

If $p^l > 0$, the right-hand term in the latter inequality converges to 0 as t goes to $+\infty$, thus $p^l = 0$, which is a contradiction. This proves that $p^l = 0$.

Finally, by definition, $0 < \mu_t = k_t^* \underline{p} \leq p_t$ for all t. The fact that $p^l = 0$ directly implies $\lim_{t\to\infty} \mu_t = \lim_{t\to\infty} k_t = 0$.

□

The rest of the proof is organized as follows. We first give a sufficient condition for μ_t to be decreasing for all $t \geq T$ for some T (Lemma 12). Then we prove that there can be only a finite number of intervals where μ_t is increasing, if any (Lemma 13).

The function $t \mapsto \mu_t$ is differentiable everywhere except at Δ, and for all $t \neq \Delta$, it holds that $\dot{\mu}_t = -\lambda\mu_t(k_t - k_{t-\Delta}\mathbb{1}_{t\geq\Delta})$. As $k_t = \mu_t/\underline{p}$, the law of motion of μ_t can be rewritten as follows:

$$\dot{\mu}_t = -\mu_t \frac{\lambda}{\underline{p}} \underbrace{(\mu_t - p_t\mu_{t-\Delta}\mathbb{1}_{t\geq\Delta})}_{:=s_t} \text{ for all } t \neq \Delta. \tag{30}$$

Let $s_t := \mu_t - p_t\mu_{t-\Delta}\mathbb{1}_{t\geq\Delta}$, so that $\dot{\mu}_t = -\mu_t s_t \lambda/\underline{p}$.

Lemma 12. *Fix $n \in \mathbb{N}$. If $\dot{\mu}_t \leq 0$ for all $t \in (n\Delta, (n+1)\Delta))$ and $s_{(n+1)\Delta} \geq 0$, then $\dot{\mu}_t \leq 0$ for all $t > (n+1)\Delta$.*

Proof. We know that, for all $t \in ((n+1)\Delta, (n+2)\Delta)$, s_t is differentiable and

$$\dot{s}_t = \dot{\mu}_t - \dot{p}_t\mu_{t-\Delta} - p_t\dot{\mu}_{t-\Delta}.$$

Suppose that there exists at least one open interval $[\underline{t}, \overline{t}] \subseteq [(n+1)\Delta, (n+2)\Delta]$ such that $\dot{\mu}_t > 0$ for all $t \in (\underline{t}, \overline{t})$. Let $[\underline{\tau}, \overline{\tau}]$ be the "smallest" of these intervals, in the sense that $\dot{\mu}_t \leq 0$ almost everywhere on $((n+1)\Delta, \underline{\tau}]$. As $s_{(n+1)\Delta} \geq 0$ by assumption, this implies that $s_t \geq 0$ almost everywhere on $[(n+1)\Delta, \underline{\tau}]$.

As $\dot{\mu}_t \leq 0$ for all $t \in (n\Delta, (n+1)\Delta))$, $\dot{\mu}_{t-\Delta} \leq 0$ for all $t \in (\underline{\tau}, \overline{\tau})$. Moreover, $\dot{p}_t \leq 0$ by definition of p_t. Therefore, for all $t \in (\underline{\tau}, \overline{\tau})$, $\dot{s}_t > 0$. As $s_{\underline{\tau}} \geq 0$, it follows that $s_t > 0$ on $[\underline{\tau}, \overline{\tau}]$, which implies $\dot{\mu}_t > 0$ on $[\underline{\tau}, \overline{\tau}]$ by (30), thus a contradiction.

As a consequence, $\dot{\mu}_t \leq 0$ almost everywhere on $((n+1)\Delta, (n+2)\Delta]$. As $\dot{\mu}_t = -\lambda\mu_t s_t$, this implies that $s_{(n+2)\Delta} \geq 0$. Iterating the argument used in the previous lines, it follows that $\dot{\mu}(t) \leq 0$ for almost all $t > (n+1)\Delta$.

□

Lemma 13. *If $\dot{\mu}_t > 0$ for all $t \in (\underline{\tau}, \overline{\tau})$, then $\overline{\tau} < +\infty$ and there is T such that $\dot{\mu}_t < 0$ for all $t \geq T$. Moreover, there is T such that $\dot{\mu}_t < 0$ for all $t \geq T$.*

Proof. Let us start with the first part of the Lemma. Suppose that $\dot{\mu}_t > 0$ for all $t \geq \underline{\tau}$. As $s_t = \mu_t - \mu_{t-\Delta}p_t > \mu_t - \mu_{t-\Delta}$, $\dot{\mu}_t > 0$ for all $t \geq \underline{\tau}$ would imply that $s_t > 0$ for all $t > \underline{\tau} + \Delta$, thus $\dot{\mu}_t < 0$, for all $t > \underline{\tau} + \Delta$, hence a contradiction. Therefore, $\overline{\tau} < +\infty$.

Let us now turn to the second part of the Lemma. Observe that, for all $t \geq \Delta$,

$$\begin{aligned}
\frac{\mu_t}{\mu_{t-\Delta}} &= \frac{x_t}{1-p_0+p_0x_{t-\Delta}} \frac{1-p_0+p_0x_{t-2\Delta}}{x_{t-\Delta}}, \\
&= \frac{1-p_0+p_0x_{t-2\Delta}}{1-p_0+p_0x_{t-\Delta}} e^{-\lambda\int_{(t-\Delta)\mathbb{1}_{t\geq\Delta}}^{t} k_s ds}, \\
&\geq e^{-\lambda\Delta\frac{p_0}{\underline{p}}},
\end{aligned}$$

since x_t is decreasing and $k_s \leq \frac{p_0}{\underline{p}}$. Rewriting (30) for $t \geq \Delta$, it holds that:

$$\dot{\mu}_t = -\frac{\lambda}{\underline{p}}\mu_t\mu_{t-\Delta}(\frac{\mu_t}{\mu_{t-\Delta}} - p_t) \leq -\frac{\lambda}{\underline{p}}\mu_t\mu_{t-\Delta}(e^{-\lambda\Delta\frac{p_0}{\underline{p}}} - p_t).$$

As $\lim_{t\to\infty} p_t = 0$, there exists $T < +\infty$ such that $p_t \leq e^{-\lambda\Delta\frac{p_0}{\underline{p}}}$ for all $t \geq T$, which implies $\dot{\mu}_t < 0$ for all $t \geq T$.

□

We can now turn to the proof of the result. For all $t < \Delta$, the expression of μ_t can be obtained by solving the differential equation $\dot{\mu}_t = -\mu_t^2\lambda/\underline{p}$ with initial condition $\mu_0 = p_0$, which yields:

$$\mu_t = \frac{p_0\underline{p}}{\underline{p} + \lambda p_0 t} \text{ for all } t < \Delta.$$

Clearly, μ_t is decreasing on $[0, \Delta]$ and, as it is continuous everywhere, $\mu_\Delta = \frac{p_0\underline{p}}{\underline{p}+\lambda p_0\Delta}$, thus $s_\Delta = \frac{p_0\underline{p}}{\underline{p}+\lambda p_0\Delta} - p_0^2$.

- If $s_\Delta \geq 0$, then $\dot{\mu}_t \leq 0$ for all $t > \Delta$ by Lemma 12.

- If $s_\Delta < 0$, because s_t is continuous, in a right neighborhood of Δ, $\dot{\mu}_t$ is well-defined and positive. Thus there exists $\underline{\tau} > \Delta$ such that $\dot{\mu}_t > 0$ on $(\Delta, \underline{\tau}]$. By Lemma 13, $\underline{\tau} < +\infty$ and there is $T > \underline{\tau}$ such that $\dot{\mu}_t < 0$ for all $t \geq T$. Hence there is a positive and finite number of intervals where μ_t is strictly increasing.

We finish by establishing the conditions under which $s_\Delta < 0$. As $s_\Delta = \frac{p_0\underline{p}}{\underline{p}+\lambda p_0\Delta} - p_0^2$, $s_\Delta < 0 \Leftrightarrow \Delta e^{-r\Delta} > \frac{\alpha(1-p_0)}{(\lambda p_0)^2}$. The function $g(\Delta) := \Delta e^{-r\Delta}$ is increasing on $[0, 1/r]$ and decreasing on $[1/r, +\infty)$, with $g(0) = \lim_{\Delta\to+\infty} g(\Delta) = 0$ and $g(1/r) = 1/(re)$.

Therefore, if $\frac{1}{re} < \frac{\alpha(1-p_0)}{(\lambda p_0)^2}$, then $s_\Delta > 0$ for all Δ. Otherwise, $s_\Delta < 0$ if and only if $\Delta \in [\underline{\Delta}_c, \overline{\Delta}_c]$, where $\underline{\Delta}_c < \frac{1}{re} < \overline{\Delta}_c$ are the two solutions of equation

$$\frac{\alpha(1-p_0)}{(\lambda p_0)^2} = \Delta e^{-r\Delta}.$$

E Proofs for Section 5.3: Endogenous investment timing

E.1 Proof of Lemma 1

Let $\mathbf{K}$ be the distribution of aggregate investment induced by some atomic equilibrium. Given that 1) players can decide when to invest and 2) individual deviations have no impact on $\mathbf{K}$, all investing players must have the same payoff in equilibrium. Thus there exists a constant $C \geq 0$ such that, for all t, if $K_t > 0$, then $P(\bar{B}_t)e^{-rt}(-\underline{p} + \nu_t(\mathbf{K})) = C$. Indeed, if there were $t \neq t'$ such that $P(\bar{B}_t)e^{-rt}(-\underline{p} + \nu_t(\mathbf{K})) > P(\bar{B}_{t'})e^{-rt'}(-\underline{p} + \nu_{t'}(\mathbf{K}))$, then each player supposed to invest at time t' would be better off investing at time t. Moreover, if $C > 0$, then all players must invest in equilibrium, because non-investing players have payoff 0. But if all players invest in equilibrium, the indifference condition $P(\bar{B}_t)e^{-rt}(-\underline{p} + \nu_t(\mathbf{K})) = C$ must hold in the limit, which implies $C = 0$. This proves that $\mathbf{K}$ satisfies

$$K_t(\nu_t(\mathbf{K}) - \underline{p}) = 0 \; \forall \, t. \tag{31}$$

This proves that any equilibrium distribution must satisfy (31).

Let us now prove by induction that there is a unique equilibrium distribution $\mathbf{K}^*$. To do so, consider two sequences $\mathbf{K}'$ and $\mathbf{K}''$ satisfying (31). We start by proving that $K'_0 = K''_0$.

- If $K'_0 = 0$, then $F(K'_0) = 1$, thus no player invests at 0 only if $p_0 \leq \underline{p}$. Yet, in that case, $F(K''_0)p_0 \leq \underline{p}$, thus $K''_0 = 0$. By symmetry, we can state that $K'_0 = 0 \Leftrightarrow K''_0 = 0$.
- If $K'_0 > 0$, then $K''_0 > 0$, thus, by (31), $F(K'_0) = F(K''_0)$, which implies $K'_0 = K''_0$ because F is strictly decreasing.

Now, fix some $t > 0$, suppose that $K'_s = K''_s$ for all $s < t$ and let us prove that $K'_t = K''_t$. Straightforwardly, under this assumption it holds that

$$\frac{\nu_t(\mathbf{K}')}{F(K'_t)} = \frac{\nu_t(\mathbf{K}'')}{F(K''_t)} \tag{32}$$

- If $K'_t = 0$, then $F(K'_t) = 1$ by definition, thus $\nu_t(\mathbf{K}') \leq \underline{p}$ since, otherwise, K'_t would be positive. Using (32), this further implies that $\nu_t(\mathbf{K}'') \leq F(K''_t)\underline{p}$. If $K''_t > 0$, then $F(K''_t) < 1$, and $\nu_t(\mathbf{K}'') < \underline{p}$, which contradicts $K''_t > 0$. Therefore, $K''_t = K'_t = 0$. As the same reasoning applies if $K''_t = 0$, this proves that $K'_t = 0 \Leftrightarrow K''_t = 0$.
- If $K'_t > 0$, then $K''_t > 0$ and $\nu_t(\mathbf{K}') = \nu_t(\mathbf{K}'') = \underline{p}$ by (31). Plugging this into (32), we obtain that $F(K'_t) = F(K''_t)$, which implies $K'_t = K''_t$ as F is strictly decreasing.

E.2 Proof of Proposition 10

Consider the sequence $I : \mathbb{N} \to \mathbb{R}$ defined by:

$$F(I_n) = \underline{p}\left(1 + \frac{1-p_0}{p_0} e^{\lambda(\sum_{m=0}^{n-1} I_k)\mathbb{1}_{n\geq 1}}\right) \; \forall \, n \in \mathbb{N}.$$

If $I_n \geq 0$, then $F(I_{n+1}) \geq \underline{p}\left(1 + \frac{1-p_0}{p_0} e^{\lambda(\sum_{m=0}^{n-1} I_k)\mathbb{1}_{n\geq 1}}\right) = F(I_n)$, which implies that $I_{n+1} \leq I_n$ as F is strictly decreasing. As, in addition, $I_0 \geq 0$ when $p_0 \geq \underline{p}$, we can state that there is $N \in \mathbb{N} \cup \{+\infty\}$, such that

$$I_n \geq 0 \text{ and } I_{n+1} \leq I_n \text{ for all } n < N.$$

Let $\bar{n}_e(p_0) := \sup\{n \in \mathbb{N},\ I_n > 0\}$ and let $\mathbf{K}$ be defined by

$$K_t = \begin{cases} I_n & \text{if } t = n\Delta \text{ and } n \le \bar{n}_e(p_0), \\ 0 & \text{otherwise.} \end{cases}$$

As we have proven that there is a unique distribution $\mathbf{K}^*$ that satisfies the equilibrium condition (14), it suffices to prove that $\mathbf{K}$ satisfies (14) to establish the "only if" part of the characterization. Let us compute $\nu_t(\mathbf{K})$ for all t. It is straightforward to prove by induction that, for all $n \in \mathbb{N}$,

$$\nu_t(\mathbf{K}) = \begin{cases} F(I_n)\dfrac{p_0 e^{-\lambda(\sum_{m=0}^{n-1} I_m)\mathbb{1}_{n\ge1}}}{1-p_0+p_0 e^{-\lambda(\sum_{m=0}^{n-1} I_m)\mathbb{1}_{n\ge1}}} & \text{if } t = n\Delta \text{ and } n \le \bar{n}_e(p_0), \\ \dfrac{p_0 e^{-\lambda(\sum_{m=0}^{n-1} I_m)\mathbb{1}_{n\ge1}}}{1-p_0+p_0 e^{-\lambda(\sum_{m=0}^{n-1} I_m)\mathbb{1}_{n\ge1}}} e^{-\lambda I_n} & \text{if } t \in (n\Delta, (n+1)\Delta) \text{ and } n \le \bar{n}_e(p_0), \\ \dfrac{p_0 e^{-\lambda \sum_{m=0}^{\bar{n}_e(p_0)} I_m}}{1-p_0+p_0 e^{-\lambda \sum_{m=0}^{\bar{n}_e(p_0)} I_m}} & \text{if } t \ge \bar{n}_e(p_0)+1. \end{cases}$$

Using the definition of $(I_n)_n$, we can rewrite $\nu_t(\mathbf{K})$ as follows:

$$\nu_t(\mathbf{K}) = \begin{cases} \underline{p} & \text{if } t = n\Delta \text{ and } n \le \bar{n}_e(p_0), \\ \frac{\underline{p}}{F(I_n)} e^{-\lambda I_n} & \text{if } t \in (n\Delta, (n+1)\Delta) \text{ and } n \le \bar{n}_e(p_0), \\ \frac{\underline{p}}{F(I_{\bar{n}_e(p_0)+1})} & \text{if } t \ge \bar{n}_e(p_0)+1. \end{cases}$$

Claim 1. *$F(I) > e^{-\lambda I}$ for all $I > 0$.*

Proof. By definition of F, $F(I) \ge e^{-\lambda I} \Leftrightarrow e^I - I - 1 \ge 0$. Yet the function $I \mapsto e^I - I - 1$ is strictly increasing on $\mathbb{R}_+$ and equals 0 when $I = 0$, which proves that it is always positive, and strictly positive if $I > 0$. □

Applying Claim 1, for all $n \in \mathbb{N}$ and all $t \in (n\Delta, (n+1)\Delta)$, $\nu_t(\mathbf{K}) < \underline{p}$ thus $K_t = 0$. Moreover, as $I_{\bar{n}_e(p_0)+1} < 0$, $\frac{\underline{p}}{F(I_{\bar{n}_e(p_0)+1})} < \underline{p}$, hence for all $t \ge \bar{n}_e(p_0)+1$, $\nu_t(\mathbf{K}) < \underline{p}$ thus $K_t = 0$. Finally, $\nu_t(\mathbf{K}) = \underline{p}$ for all $t = n\Delta$, with $n \le \bar{n}_e(p_0)$. Therefore, $\mathbf{K}$ satisfies (31), which proves the "only if" part of the result.

We now turn to the "if" part. Consider an atomic strategy profile inducing $\mathbf{K}^*$. Let us prove that this profile is an equilibrium. Fix any player i. If player invests i at time $t = n\Delta$, with $n \le \bar{n}_e(p_0)$, then she obtains 0, by definition of $\mathbf{K}^*$. If she invests at any other t, i.e., between atoms or after $\bar{n}_e(p_0)$, she obtains a strictly negative payoff, also by definition of $\mathbf{K}^*$ and $\bar{n}_e(p_0)$.

E.3 Proof of Proposition 11

By definition of $\bar{n}_e$, $\bar{n}_e \ge 1 \Leftrightarrow I_1 > 0 \Leftrightarrow F(I_1) < 1$. By definition of F, $F(I_1) < 1$ if and only if

$$H(p_0) := p_0 - \underline{p} - \underline{p}(1-\underline{p})g(p_0) > 0, \tag{33}$$

with $g(p_0)$ determined by $\frac{1-e^{-g(p_0)}}{g(p_0)} = \frac{\underline{p}}{p_0}$. As $g(\underline{p}) = 0$ and $1 - \underline{p}g(1) = e^{-g(1)}$, we have $H(\underline{p}) = 0$ and $H(1) = (1-\underline{p})e^{-g(1)} > 0$. We start by proving the following Lemma:

Lemma 14. *Let $g(p_0)$ be defined by $F(\frac{g(p_0)}{\lambda}) = \frac{\underline{p}}{p_0}$, i.e.,*

$$\frac{1-e^{-g(p_0)}}{g(p_0)} = \frac{\underline{p}}{p_0}. \tag{34}$$

It holds that

(i) $g'(p_0) > 0$ *for all* $p_0 \in [\underline{p}, 1)$;

(ii) $\lim_{p_0 \to \underline{p}} g'(p_0) = \frac{2}{\underline{p}}$;

(iii) $g''(p_0) < 0$ *for all* $p_0 \in [\underline{p}, 1)$.

(iv) $\frac{\partial g(p_0)}{\partial \underline{p}} = -\frac{g(p_0)}{\underline{p} - p_0 + \underline{p}g(p_0)}$

Proof. *(i)* Differentiating (34) with respect to p_0 and rearranging, we obtain

$$g'(p_0)(\underline{p} - p_0 e^{-g(p_0)}) = 1 - e^{-g(p_0)} \tag{35}$$

Let us prove that $\underline{p} - p_0 e^{-g(p_0)} > 0$. We rewrite $\underline{p} - p_0 e^{-g(p_0)} = p_0(\frac{\underline{p}}{p_0} - e^{-g(p_0)}) = p_0(F(g(p_0)) - e^{-g(p_0)})$. Now, consider $h(x) := F(x) - e^{-x}$. Clearly, $h(0) = 0$. Moreover, differentiating h with respect to x, we obtain $h'(x) > 0$ for all $x > 0$. This implies $h(x) > 0$ for all $x > 0$, thus $g'(p_0) > 0$.

(ii) As $g'(p_0) = \frac{1-e^{-g(p_0)}}{\underline{p} - p_0 e^{-g(p_0)}}$ and $g(\underline{p}) = 0$, we use L'Hôpital's rule to find that $\lim_{p_0 \to \underline{p}} g'(p_0) = 2/\underline{p}$.

(iii) Differentiating (35) with respect to p_0, we find that

$$g''(p_0)(\underline{p} - p_0 e^{-g(p_0)}) = g'(p_0)e^{-g(p_0)}(2 - p_0 g'(p_0))$$

As $(\underline{p} - p_0 e^{-g(p_0)})$ and $g'(p_0)$ are both positive, the sign of $g''(p_0)$ is the sign of $(2 - p_0 g'(p_0))$. Let us prove that $g'(p_0) > 2/p_0$ almost everywhere on $(\underline{p}, 1]$. Suppose in contrast that there are $\underline{p} < x < y < 1$ such that $g'(p_0) < 2/p_0$ for all $p_0 \in [x, y]$. As it implies that $g''(p_0) > 0$ on $[x, y]$, $g'(x) < 2/x < 2/\underline{p} = g'(\underline{p})$. As g' is continuous, this further implies that g' is decreasing on some interval $[x', y'] \subset [\underline{p}, x]$ with $g'(p_0) < 2/p_0$ on this interval, which is impossible.

(iv) Differentiating (34) with respect to $\underline{p}$, we obtain

$$-\frac{\partial g(p_0)}{\partial \underline{p}} p_0 e^{-g(p_0)} = -g(p_0) - \underline{p}\frac{\partial g(p_0)}{\partial \underline{p}}$$

Rearranging and using (34), we obtain the desired expression.

□

Differentiating $H(p_0)$ twice with respect to p_0, we obtain $H'(p_0) = 1 - \underline{p}(1-\underline{p})g'(p_0)$ and $H''(p_0) = -\underline{p}(1-\underline{p})g''(p_0)$. By Lemma 14, H is convex on $[\underline{p}, 1]$, which implies that $H'(p_0)$ increases on $[\underline{p}, 1]$. Invoking again Lemma 14, $\lim_{p_0 \to \underline{p}} H'(p_0) = 2\underline{p} - 1$. Therefore,

- if $\underline{p} \geq 1/2$, then $H(p_0) > 0$ for all $p_0 \in [\underline{p}, 1]$;
- if $\underline{p} < 1/2$, then there is $x \in (\underline{p}, 1)$ such that $H'(p_0) < 0 \Leftrightarrow p_0 \in [\underline{p}, x)$, which further implies that there exists a unique $\tilde{p}(\underline{p}) \in (x, 1)$, defined by $H(\tilde{p}(\underline{p})) = 0$ such that $H(p_0) > 0$ if and only if $p_0 > \tilde{p}(\underline{p})$.

As a consequence, $H(p_0) > 0$ if and only if (a) or (b) holds, with

(a) $\underline{p} \geq 1/2$;

(b) $\underline{p} < 1/2$ and $p_0 \geq \tilde{p}(\underline{p})$;

The next Lemma states that $\tilde{p}(\underline{p})$ decreases with $\underline{p}$ on $[0, 1/2]$.

Lemma 15. $\frac{\partial \tilde{p}(\underline{p})}{\partial \underline{p}} < 0$ *for all* $\underline{p} \in [0, 1/2)$.

Proof. For the sake of clarity, let $h(p_0, \underline{p}) := H(p_0) = p_0 - \underline{p} - \underline{p}(1-\underline{p})g(p_0)$. By definition of $\tilde{p}(\underline{p})$, $h(\tilde{p}(\underline{p}), \underline{p}) = 0$, hence

$$\tilde{p}(\underline{p}) - \underline{p} - \underline{p}(1-\underline{p})g(\tilde{p}(\underline{p})) = 0. \tag{36}$$

Differentiating $h(\tilde{p}(\underline{p}), \underline{p}))$ with respect to $\underline{p}$, we find that

$$h_1(\tilde{p}(\underline{p}), \underline{p})\tilde{p}'(\underline{p}) + h_2(\tilde{p}(\underline{p}), \underline{p}) = 0.$$

By definition, $h_1(\tilde{p}(\underline{p}), \underline{p})) = H'(\tilde{p}(\underline{p}))$, which is positive as $\tilde{p}(\underline{p}) > x$. Let us prove that $h_2(\tilde{p}(\underline{p}), \underline{p})$ is also positive. Differentiating $h(p_0, \underline{p})$ with respect to $\underline{p}$, we obtain:

$$h_2(p_0, \underline{p}) = -1 + (2\underline{p} - 1)g(p_0) - \underline{p}(1-\underline{p})\frac{\partial g(p_0)}{\partial \underline{p}}.$$

By (iv) in Lemma 14, $\frac{\partial g(p_0)}{\partial \underline{p}} = -\frac{g(p_0)}{\underline{p} - p_0 + \underline{p}g(p_0)}$. Therefore,

$$\frac{\partial g(\tilde{p}(\underline{p}))}{\partial \underline{p}} = -\frac{g(\tilde{p}(\underline{p}))}{\underline{p} - \tilde{p}(\underline{p}) + \underline{p}g(\tilde{p}(\underline{p}))}.$$

Plugging (36) into the latter expression, we obtain:

$$\frac{\partial g(\tilde{p}(\underline{p}))}{\partial \underline{p}} = -\frac{1}{\underline{p}^2}.$$

Therefore,

$$h_2(\tilde{p}(\underline{p}), \underline{p}) = -1 + (2\underline{p} - 1)g(\tilde{p}(\underline{p})) + \frac{1-\underline{p}}{\underline{p}} = \frac{1-2\underline{p}}{\underline{p}}(1 - \underline{p}g(\tilde{p}(\underline{p}))),$$

which is positive because $\underline{p} < 1/2$ and $1 - \underline{p}g(p_0) > p_0 - \underline{p}g(p_0) = p_0 e^{-g(p_0)} > 0$. □

We can now conclude the proof. First, as $\tilde{p}(\underline{p})$ is decreasing, $p_0 \geq \tilde{p}(\underline{p}) \Leftrightarrow \underline{p} > \tilde{p}^{-1}(p_0) \Leftrightarrow \Delta > \underline{\Delta}_e(p_0)$ where $\underline{\Delta}_e(p_0)$ solves $p_0 = \tilde{p}(\underline{p})$, i.e., $F(I_1) = 1$. As, in addition, $\tilde{p}(1/2) = 1/2$, we know that $\tilde{p}(\underline{p}) > 1/2$ thus $p_0 > \tilde{p}(\underline{p}) \Rightarrow p_0 > 1/2$. Therefore,

$$(b) \Rightarrow p_0 > 1/2 \text{ and } \Delta > \underline{\Delta}_e(p_0).$$

Second, $\underline{p} \geq 1/2$ implies that $\Delta > \overline{\Delta}_e := \frac{1}{r} \ln\left(\frac{\lambda}{2\alpha}\right)$. It also implies that $p_0 > 1/2$ since $p_0 > \underline{p}$. Moreover, it is straightforward to show that, if $p_0 > 1/2$, then $\overline{\Delta}_e > \underline{\Delta}_e(p_0)$. Therefore,

$$(a) \Rightarrow p_0 > 1/2 \text{ and } \Delta > \underline{\Delta}_e(p_0).$$

This proves that $H(p_0) > 0$ only if $p_0 > 1/2$ and $\Delta > \underline{\Delta}_e(p_0)$. The "if" implication is straightforward.